UCLA School of Engineering and Applied Science

A STUDY OF STRONG STABILITY OF DISTRIBUTED SYSTEMS

December 1989

Tayfun Cataltepe

UCLA SCHOOL OF ENGINEERING AND APPLIED SCIENCE

UNIVERSITY OF CALIFORNIA

Los Angeles

A STUDY OF STRONG STABILITY OF DISTRIBUTED SYSTEMS

A dissertation submitted in partial satisfaction of the

requirements for the degree Doctor of Philosophy

in Electrical Engineering

by

Tayfun Cataltepe

1989

The dissertation of Tayfun Cataltepe is approved.

Donald Babbitt

James S. Gibson

Paul K. C. Wang

Kung Yao

Nhan Levan, Committee Chair

University of California, Los Angeles

1989

To my parents Keriman and Mürşit Çataltepe

Contents

ACKNOWLEDGEMENTS

I would like to thank my advisor Professor Nhan Levan for his constant support and friendship throughout my graduate work at UCLA. Without his guidance none of this would have been possible. I would also like to thank my dissertation committee members Professor Donald Babbitt, Professor James S. Gibson, Professor Paul K. C. Wang and Professor Kung Yao.

This research has been supported by the United States Air Force Office of Scientific Research, Directorate of Mathematical and Information Sciences under Grant AFOSR 86-0132, and by NASA Ames-Dryden under Grant NAG 2-303.

VITA

Tayfun Cataltepe

[redacted]	Born, [redacted]
1982	B.Sc., Electrical Engineering Bogazici University Istanbul, Turkey
1984	M.Sc., Electrical Engineering Michigan Technological University Houghton, Michigan

PUBLICATIONS

Cataltepe,T., Stanek, E. K., Wiitanen, D.O., Yenchek, M.R., *Performance and test criteria for DC fuses in coal mine applications*, IEEE Tran. on Industry Appl., pp 352-359, March/April 1986.

Cataltepe, T., Wiitanen, D.O., Stanek, E.K., *Modeling of transients in a three phase uncontrolled bridge rectifier*, Midwest Power Symp. Proc., Philadelphia, PA, October 1984.

Cataltepe. T., Rogers, J.C., Stanek, E.K., VanVeen, B., Wiitanen, D.O., *Characteristics of DC mine power systems affecting fuse performance*, Conference Records, IEEE Industry Appl. Soc. (IAS) Mining Industry Technical Conf., Houghton, MI, pp 85-102, June 1983.

Stanek, E.K., Cataltepe, T., Wiitanen, D.O., *Phenomena that affect the calculation of the inductance and resistance of mine track/trolley systems*, Proc. of IEEE IAS Annual Meeting, Chicago, IL, pp 100-106, October 1984.

ABSTRACT OF THE DISSERTATION

A STUDY OF STRONG STABILITY OF DISTRIBUTED SYSTEMS

by

Tayfun Cataltepe

Doctor of Philosophy in Electrical Engineering

University of California, Los Angeles, 1989

Professor Nhan Levan, Chair

This dissertation studies strong stability of distributed systems and addresses the problem of characterizing strongly stable semigroups of operators associated with distributed systems. Main emphasis is on contractive systems. Three different approaches to characterization of strongly stable contractive semigroups are developed. First one is an operator theoretic approach. Using the theory of dilations it is shown that every strongly stable contractive semigroup is related to the left shift semigroup on an L^2 space. Then, a decomposition for the state space which identifies strongly stable and unstable states is introduced. Based on this decomposition, conditions for a contractive semigroup to be strongly stable are obtained. Finally, extensions of Lyapunov equation for distributed systems are investigated. Sufficient conditions for weak and strong stabilities of uniformly bounded semigroups

are obtained by relaxing the equivalent norm condition on the right hand side of the Lyapunov equation.

These characterizations are then applied to the problem of feedback stabilization. First, it is shown via the state space decomposition that under certain conditions a contractive system (A, B) can be strongly stabilized by the feedback $-B^*$. Then, application of the extensions of the Lyapunov equation results in sufficient conditions for weak, strong and exponential stabilizations of contractive systems by the feedback $-B^*$. Finally, it is shown that for a contractive system $\dot{x} = Ax + Bu$, where B is any linear bounded operator, there is a related linear quadratic regulator problem and a corresponding steady state Riccati equation which always has a bounded nonnegative solution.

Chapter 1

Introduction

Distributed systems are those dynamical systems that are described by partial differential equations, integral equations, integro-differential equations or delay-differential equations. They differ from lumped parameter systems in that the state space of the system is no longer finite dimensional.

A flexible beam, an electrical transmission line, temperature distribution on a metal rod, evolution of the population of a country, e.t.c., are all examples of distributed systems.

A stable dynamical system is one in which the state does not grow without bound in time. Topology of the state space plays the most important role in defining the stability of the system. For distributed systems this leads to three different types of stabilities: weak, strong, and exponential. A distributed system is weakly stable if the state converges to zero in time in the weak topology and it is strongly stable if the convergence is in the strong topology. When the norm of the state decays exponentially the system is exponentially stable. Exponential stability is equivalent to stability in the uniform operator topology.

Given a system, the problem of choosing a control (or forcing function) that will yield the desired degree of stability of the state is referred to as stabilization. Stability and stabilizability are of primary interest in control theory.

In this dissertation, we study strong stability. Specifically, we find conditions for a linear distributed system to be strongly stable and strongly stabilizable. There are various reasons for concentrating on strong stability rather than weak or exponential stabilities. Exponential stability is a more desirable property, however it is difficult to come by in many situations [4, 7]. For example, a strongly stable distributed system cannot be exponentially stabilized by a compact feedback. This means that no finite dimensional controller can enhance the stability of the system [22, 68]. Another situation is that exact controllability, a property which guarantees the existence of an exponentially stabilizing control, is not possible if the system or the control operator is compact. Certain parabolic systems and delay differential equations are examples of such systems [65, 67]. In these cases strong stability may play a useful role [4, 22].

Strong stability involves the state space. In other words, some states may be strongly stable some may not. By the same token, it may be possible to strongly stabilize a subset of the state space.

Moreover, since a norm can be regarded as energy, strong stability may be interpreted as energy being dissipated in time while weak stability in general, does not have such a physical meaning.

Most of the previous work on stability and stabilizability of distributed systems deals with exponential and weak stabilities [7, 12, 13, 62]. Necessary and sufficient conditions for exponential stability are known. However, there are no general

conditions for weak or strong stabilities. These are the central problems of our work.

This dissertation is organized as follows. Chapter 2 reviews the essential mathematical notions and results on distributed systems. First, we state the fundamental definitions and theorems on strongly continuous semigroups that will be needed in subsequent chapters. Then we introduce the abstract state space model for linear distributed systems and give definitions of controllability, stability and stabilizability. Finally, results on stability and stabilizability of distributed systems that are relevant to our study are stated.

In Chapter 3, three different approaches to characterization of strongly stable (mainly contractive) semigroups are developed. First, we apply the theory of dilations of contractions to strongly stable contractive semigroups. It turns out that every strongly stable contractive semigroup is "related" to the left shift semigroup on an L^2 space. In Section 3.2, we introduce a decomposition for the state space which identifies strongly stable and unstable states of a contractive system. This decomposition then results in conditions for strong stability. Finally we investigate extensions of Lyapunov equation for weak and strong stabilities of uniformly bounded semigroups.

Chapter 4 is an application of the results of Chapter 3 for feedback stabilization. First, we apply the state space decomposition developed in Section 3.2 to a contractive system (A, B) with feedback $-B^*$. This results in conditions for strong stabilization via the feedback $-B^*$. Then we apply the two extensions of the Lyapunov equation investigated in Section 3.3. Finally we show that the application of the Lyapunov equation leads to some results concerning a linear quadratic regulator

problem for contractive systems.

Chapter 5 contains a discussion of our results and ideas for future research directions.

Chapter 2

Background and Review

In this chapter we review basic mathematical fundamentals relating to linear distributed systems and define the terminology that will be used throughout this work. We also present the key results on stability and stabilizability of distributed systems which are pertinent to our study. However, not all of the related results will be stated here, instead, we will refer to them in the subsequent chapters as needed.

The abstract differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \quad ,t > 0 \\ x(0) &= x_0 \end{aligned} \tag{2.1}$$

serves as a model for linear time invariant control systems. For each t, $x(t)$ evolving according to equation (2.1) is called the state at time t and belongs to a Hilbert space H called the state space. $u(t)$, called the control, belongs to a second Hilbert space U, the control space. A and B are linear operators. When H and U are finite dimensional, (2.1) becomes the familiar state space model for lumped parameter linear control systems. If the state space H is infinite dimensional equation (2.1)

describes a distributed control system. The operator A is taken to be the generator of a strongly continuous semigroup of linear bounded operators $T(t)$, $t \geq 0$. Properties of the spaces H and U, and those of the operators A (hence the semigroup $T(t)$) and B determine the characteristics of the control system such as controllability and stability.

2.1 Strongly Continuous Semigroups

The theory of semigroups of linear operators provides a convenient setting for studying linear systems. A family of linear bounded operators $T(t)$, $t \geq 0$, over a Hilbert space H is called a strongly continuous (C_0) semigroup if

(i) $T(t)T(s) = T(t+s)$ for $t, s \geq 0$,

(ii) $T(0)x = x$ for all $x \in H$,

(iii) $t \to T(t)x$ is a continuous mapping for $t \geq 0$ and each $x \in H$.

The linear operator A defined by

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}$$

is called the infinitesimal generator of the semigroup $T(t)$. The domain $D(A)$ of A consists of all those x in H for which the limit exists. We have:

Theorem 2.1.1 [3, 29] *A linear operator A is the generator of a C_0 semigroup $T(t)$ on H if and only if*

(i) *A is closed and $D(A)$ is dense in H,*

(ii) *There exist $M \geq 1$ and $\omega \in \Re$ (real) such that for each $\lambda > \omega$, $\lambda \in \rho(A)$ -resolvent set of A- and for $n \geq 0$*

$$\|(\lambda - \omega)^n (\lambda I - A)^{-n}\| \leq M .$$

Moreover $\|T(t)\| \leq M e^{\omega t}$.

The resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which the operator $(\lambda I - A)$ has a bounded inverse. The family of linear bounded operators

$$R_\lambda(A) = (\lambda I - A)^{-1} \quad , \lambda \in \rho(A) ,$$

is called the resolvent of A. The complement of the resolvent set in the complex plane is the spectrum of A and is denoted by $\sigma(A)$. Eigenvalues, or the point spectrum, form a subset of $\sigma(A)$. If $\|T(t)\| \leq M e^{\omega t}$ for some ω then the spectrum of the generator A is contained in the halfplane $\{\lambda : \mathrm{Re}.\lambda \leq \omega\}$.

If A generates $T(t)$, then its adjoint A^* generates $T(t)^*$, which is also a C_0 semigroup. The restriction to an invariant subspace of a semigroup and the direct sum of semigroups are again semigroups.

Theorem 2.1.2 [29] *Let A be the generator of a C_0 semigroup $T(t)$ and let B be a linear bounded operator on the space H. Then $A + B$ also generates a C_0 semigroup $S(t)$ which satisfies*

$$S(t)x = T(t)x + \int_0^t T(t - \sigma) B S(\sigma) \, d\sigma \quad , x \in H.$$

Moreover, if $\|T(t)\| \leq M e^{\omega t}$ then $\|S(t)\| \leq M e^{(\omega + M\|B\|)t}$.

A C_0 semigroup $T(t)$ is said to be compact if $T(t)$ is compact for each $t > 0$. $T(t)$ is called uniformly bounded if $\|T(t)\| \leq M$ for some $M \geq 1$ and for all $t \geq 0$. It is called a contraction or contractive semigroup when $\|T(t)\| \leq 1$.

Theorem 2.1.3 [29] *An operator A is the generator of a contractive semigroup if and only if it is maximal dissipative, that is,*

$$Re.\,[Ax, x] \leq 0 \ \textit{for all} \ x \in D(A)$$

and A does not admit any proper dissipative extension.

For a contraction semigroup $T(t)$, $t \geq 0$, the following are well defined closed subspaces:

(i) Isometric subspace

$$H_i(T) = \{x \in H : \|T(t)x\| = \|x\| \ \text{for} \ t \geq 0\},$$

(ii) Unitary subspace:

$$H_u(T) = H_i(T) \bigcap H_i(T^*) \ .$$

It is clear that $H_i(T)$ is invariant under $T(t)$ and $H_u(T)$ reduces $T(t)$, $t \geq 0$. The contraction semigroup $T(t)$ is called completely non-isometric (c.n.i.) if $H_i(T) = \{0\}$ and is completely non-unitary (c.n.u.) if $H_u(T) = \{0\}$. Since, by definition, $H_u(T) \subseteq H_i(T)$, it follows that c.n.i. $\Rightarrow$ c.n.u. .

For a detailed treatment of the theory of semigroups, we refer to [3, 14, 23, 27, 55, 70].

The following terminology will be used in this work:

Definition 2.1.1 *A linear bounded self-adjoint operator P is*

(i) *nonnegative if* $0 \leq [Px, x]$ *for all $x \in H$,*

(ii) *positive if* $0 < [Px, x]$ *for all $x \in H$,*

(iii) *strictly positive if* $\alpha\|x\|^2 \leq [Px, x]$ *for some* $\alpha > 0$ *and for all* $x \in H$.

Example 2.1.1 Let K be a Hilbert space and denote by $L^2([0,\infty); K)$ the space of K-valued functions $x(t)$ such that

$$\|x\|_{L^2}^2 = \int_0^\infty \|x(t)\|_K^2 dt < \infty .$$

Then, $L^2([0,\infty); K)$ equipped with the inner product

$$[x, y]_{L^2} = \int_0^\infty [x(t), y(t)]_K dt$$

is a Hilbert space.

The family of operators $L(t)$, $t \geq 0$, on $L^2([0,\infty); K)$ defined by

$$(L(t)x)(s) = x(s+t)$$

form a C_0 contractive semigroup since $\|L(t)x\|_{L^2} \leq \|x\|_{L^2}$. $L(t)$ is called the *left shift semigroup*.

A simple calculation shows that the generator of the left shift semigroup is

$$(Ax)(t) = \frac{dx}{dt}$$

with

$$D(A) = \{x : x \text{ absolutely continuous}, \frac{dx}{dt} \in L^2([0,\infty); K), x(0) = 0\} .$$

The adjoint of $L(t)$ is given by

$$(R(t)x)(s) = (L(t)^* x)(s) = \begin{cases} 0 & , 0 \leq s < t \\ x(s-t) & , t \leq s \end{cases}$$

and it is called the *right shift semigroup*. Since $\|R(t)x\| = \|x\|$ for all x, $R(t)$ is a semigroup of isometries. We have $\|R(t)\| = \|L(t)\| = 1$, for $t \geq 0$. $\triangle$

2.2 Linear Distributed Systems

Consider the abstract homogeneous differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) \ , t > 0 \\ x(0) &= x_0 \end{aligned} \tag{2.2}$$

where for each $t \geq 0$, $x(t)$ is an element of an infinite dimensional Hilbert space H and A is the generator of a C_0 semigroup of linear bounded operators on H. Under these conditions, (2.2) represents a linear autonomous distributed system. If $T(t)$, $t \geq 0$, is the semigroup generated by A, then the unique (mild) solution of (2.2) is given by $x(t) = T(t)x_0$.

The non-homogeneous equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \ , \\ x(0) &= x_0 \ , \end{aligned} \tag{2.3}$$

describes a linear distributed control system and is denoted by the pair (A, B). The control $u(t)$ takes values in the space U and, for our purposes, we take $u(\cdot) \in L^2([0, \infty); U)$ and B to be a linear bounded operator from U to H. In this case, the (mild) solution of (2.3) is of the form

$$x(t) = T(t)x_0 + \int_0^t T(t - \sigma)Bu(\sigma)d\sigma \ . \tag{2.4}$$

We say that the system (A, B) is *contractive* if A generates a contraction semigroup, and *conservative* if A generates an isometric semigroup.

2.2.1 Controllability of Linear Distributed Systems

A state $x \in H$ is said to be *reachable* (from origin) if there exist $t > 0$ and $u(\cdot) \in L^2([0,t);U)$ such that

$$x = \int_0^t T(t-\sigma)Bu(\sigma)d\sigma \ .$$

The characteristic of the system (2.3) that allows the control $u(t)$ to steer the state from one point to another in the state space is referred to as controllability. There are various notions of controllability for distributed systems, all of which are equivalent if the state space is finite dimensional [3, 7, 56, 63].

Definition 2.2.1 *The system (A,B) is exactly controllable if for every $x \in H$, there exist $t > 0$ and $u(\cdot) \in L^2([0,\infty);U)$ such that*

$$x = \int_0^t T(t-\sigma)Bu(\sigma)d\sigma \ .$$

The above definition is a natural extension of the concept of controllability in finite dimensional systems. However, exact controllability is difficult to achieve and many real life distributed systems are not exactly controllable [3]. For example if the operator B or the semigroup $T(t)$ is compact, then (A,B) is never exactly controllable unless the state space is finite dimensional. In this work we will be concentrating on the following weaker notion of controllability [6, 40]:

Definition 2.2.2 *A state x in H is called (approximately) controllable if for $\epsilon > 0$, there is a $u(\cdot) \in L^2([0,t);U)$ such that*

$$\|x - \int_0^t T(t-\sigma)Bu(\sigma)d\sigma\| < \epsilon \ \textit{for some} \ t > 0 \ .$$

The set of all (approximately) controllable states is

$$M_c(A,B) = \overline{\bigcup_{t\geq 0} T(t)BU}$$

where $\overline{\cdot}$ denotes the closure. The orthogonal complement in H of M_c, denoted by M_{uc}, is called the (approximately) uncontrollable subspace:

$$M_{uc}(A, B) = \bigcap_{t \geq 0} \mathbf{N}(B^* T(t)^*) \ .$$

The system (A, B) is (approximately) controllable if and only if $M_{uc}(A, B) = \{0\}$ $\Longleftrightarrow M_c(A, B) = H$.

It follows that a system (A, B) is exactly controllable if every state in H is reachable, and it is approximately controllable if the set of reachable states is dense in H.

From now on, we will refer to "approximate controllability" simply as "controllability". It is easy to see that a state $x \in H$ is uncontrollable if $B^*T(t)^*x = 0$ for $t \geq 0$ and the system (A, B) is controllable if and only if

$$B^*T(t)^*x = 0 \quad \text{for} \quad t \geq 0 \Longrightarrow x = 0 \ \ .$$

The following theorem shows that controllability of a distributed system is unaffected by bounded perturbations.

Theorem 2.2.1 *Let F be a linear bounded operator. Then (A, B) controllable $\Longleftrightarrow (A + F, B)$ controllable.*

2.2.2 Stability of Linear Distributed Systems

In control theory, stability of a system is of foremost importance. The autonomous system (2.2) is stable if for every initial state x_0 in H

$$x(t) \to 0 \ \text{ as } \ t \to \infty \ .$$

Equivalently, a state $x_0 \in H$ is stable if

$$T(t)x_0 \to 0 \text{ as } t \to \infty . \tag{2.5}$$

Hence, stability of the semigroup $T(t)$ determines stability of the system. In finite dimensional state space, (2.5) implies that the decay is exponential. However, in infinite dimensional state space, we have the following notions of stability:

Definition 2.2.3 *The system (2.2) or equivalently, the semigroup $T(t)$, $t \geq 0$, is*

(i) *exponentially (e)-stable if there are constants $M \geq 1$ and $\omega > 0$ such that*

$$\|T(t)\| \leq Me^{-\omega t} \quad , t \geq 0 ,$$

(ii) *strongly (s)-stable if, for every x in H*

$$\|T(t)x\| \to 0 \quad as \quad t \to \infty ,$$

(iii) *weakly (w)-stable if, for all x and y in H*

$$[T(t)x, y] \to 0 \quad as \quad t \to \infty .$$

It is clear that e-stability $\Rightarrow$ s-stability $\Rightarrow$ w-stability. Also, by the uniform boundedness principle, a stable semigroup is always uniformly bounded. We say that the system (A, B) is stable if the semigroup generated by A is stable.

Note that if a semigroup $T(t)$ is weakly or exponentially stable then so is its adjoint $T(t)^*$. This is not necessarily true for strong stability. For example, the left shift semigroup $L(t)$ of Example 2.1.1 is strongly stable but its adjoint $R(t)$ (right shift) is not since the right shift semigroup is isometric.

We now review the pertinent results on stability of C_0 semigroups. Almost all of the previously reported works on weak and strong stabilities dealt with contraction semigroups. This, as we shall see, is due to the fact that such semigroups have certain special properties that can be exploited to arrive at conditions for weak and strong stabilities. One of the most useful properties of contractions is the following decomposition due to Sz.-Nagy and Foias:

Theorem 2.2.2 [51] *To every contractive semigroup $T(t)$, $t \geq 0$, on H, there corresponds a unique orthogonal decomposition*

$$H = H_{cnu}(T) \oplus H_u(T)$$

where $H_u(T)$ is the maximal subspace which reduces $T(t)$ to a unitary semigroup while its orthogonal complement $H_{cnu}(T)$ reduces the semigroup to a completely non-unitary contractive semigroup. $T(t)$ admits the unique canonical decomposition

$$T(t) = T_{cnu}(t) \oplus T_u(t) \quad .$$

What makes the above decomposition applicable to the study of stability of contractions is Foguel's characterization of the weakly unstable states.

Theorem 2.2.3 [20] *Let $T(t)$, $t \geq 0$, be a contractive semigroup on H. Let*

$$W(T) = \{x \in H : T(t)x \to 0 \ \ \textit{weakly as} \ \ t \to \infty\} \ .$$

Then, $W(T)$ is a closed subspace which reduces $T(t)$. Moreover, $W(T) = W(T^*)$ and $W(T)^\perp \subseteq H_u(T)$.

$W(T)$ can be called the weakly stable subspace. Its orthogonal complement $W(T)^\perp$ is therefore called the weakly unstable subspace. A contraction semigroup

$T(t)$, then, is weakly stable if and only if $W(T)^{\perp} = \{0\}$. Foguel's theorem shows that a completely non-unitary contraction semigroup is weakly stable. Note that this is only a sufficient condition.

Analogous to $W(T)$, we can define $M_s(T)$ to be the strongly stable subspace of a semigroup $T(t)$:

$$M_s(T) = \{x \in H : \|T(t)x\| \to 0 \text{ as } t \to \infty\} .$$

There are no general conditions for a C_0 semigroup to be weakly or strongly stable. However, under certain situations, weak and strong stabilities are equivalent. Fortunately, most physical systems have these properties [3, 23].

Theorem 2.2.4 [5] *Let $T(t)$, $t \geq 0$, be a C_0 semigroup. Weak stability implies strong stability provided, either*

(i) *The generator A has a compact resolvent,*

or

(ii) *$T(t)$ is self-adjoint.*

Recently, Levan [10, 43, 45, 46] and Miyaji [48] investigated strong stability of C_0 semigroups via several different approaches. We will be referring to their work later on.

The strongest type of stability is exponential stability. First, we have:

Theorem 2.2.5 [3] *If the semigroup $T(t)$ is compact for some $t > 0$, then weak stability implies exponential stability.*

Necessary and sufficient conditions for a C_0 semigroup to be exponentially stable are known. The following theorem which can be regarded as the infinite dimensional analog of Lyapunov's theorem on stability of matrices, was proven by Datko:

Theorem 2.2.6 [12] *Let $T(t)$, $t \geq 0$, be a C_0 semigroup with generator A in H. The following statements are equivalent:*

(i) *$T(t)$ is exponentially stable.*

(ii) *There exists a self-adjoint, positive operator $P > 0$ on H satisfying*

$$2 \text{ Re. } [PAx, x] = -\|x\|^2$$

for all x in the domain $D(A)$ of A.

(iii) *For all x in H,*

$$\int_0^\infty \|T(t)x\|^2 dt < \infty \ .$$

Condition (ii) may be replaced by

$$2 \text{ Re. } [PAx, x] = -[Wx, x] \ ,$$

where W is a self-adjoint, strictly positive operator, i.e., $[Wx, x] \geq \alpha\|x\|^2$ for some $\alpha > 0$. In this case, W defines an equivalent norm.

The third condition of the theorem was generalized by Pazy:

Theorem 2.2.7 [54] *A C_0 semigroup $T(t)$, $t \geq 0$, on a Banach space is exponentially stable if and only if for $1 \leq p \leq \infty$*

$$\|x\|_p = (\int_0^\infty \|T(t)x\|^p dt)^{1/p} < \infty \ .$$

Moreover, $\|\cdot\|_p$ *defines an equivalent norm if and only if there exists* $t_0 > 0$ *and* $c > 0$ *such that, for every* x,

$$\|T(t_0)x\| \geq c\|x\| \, .$$

2.2.3 Stabilizability of Linear Distributed Systems

Suppose the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is not stable in any sense. Feedback stabilization or stabilization in short, is the problem of finding a linear feedback operator $F : H \to U$ so that the system with the feedback control $u(t) = Fx(t)$ is stable. The system (A, B) is said to be weakly, strongly or exponentially stabilizable if such an operator F exists that weakly, strongly or exponentially stabilizes the system.

Thus, stabilization of the system (2.3) is equivalent to the stability of the following system with the perturbed generator:

$$\begin{aligned} \dot{x}(t) &= (A + BF)x(t) \\ x(0) &= x_0 \, . \end{aligned} \tag{2.6}$$

The solution of this differential equation is

$$x(t) = S(t)x_0 \quad , t \geq 0$$

where $S(t)$, $t \geq 0$, is the C_0 semigroup generated by $A + BF$. The stabilization problem, then, can be restated as: given a semigroup $T(t)$ with generator A on H, find an operator $F : H \to U$ such that the semigroup $S(t)$ generated by $A + BF$ is stable.

Benchimol proved the following, which is a refinement of earlier results on weak stabilization of contraction semigroups [52, 53]:

Theorem 2.2.8 [5] *Let A be the generator of a contraction semigroup $T(t)$ on H, and B a linear bounded operator from another Hilbert space U into H. Then the semigroup $S(t)$ generated by $A - BB^*$ is weakly stable if and only if the weakly unstable subspace of $T(t)$ is (approximately) controllable.*

Another way of stating the above theorem is that the contractive system $\dot{x} = Ax + Bu$ is weakly stabilizable by the control $u(t) = -B^*x(t)$ if and only if $W(T)^{\perp} \subseteq M_c(A, B)$. This choice of control is robust in the sense that it does depend on the characteristics of the uncontrolled system. The result also applies to semigroups which are similar to contractions.

Other types of feedback controls for weak stabilization of both general C_0 semigroups and quasi-affine transforms of contraction semigroups have been studied by Levan [43]. Levan showed that using a feedback involving a solution of the steady state Riccati equation, a C_0 semigroup can be approximately weakly stabilized. The stabilization is approximate in the sense that the feedback semigroup is stable on a dense subspace instead of on the whole space [44].

When the generator of the original semigroup $T(t)$ has compact resolvent, weak and strong stabilities are equivalent. Indeed, most of the results on strong stabilization are derived for systems with compact resolvent [4, 6]. However, using state space decomposition techniques, Levan and Rigby gave the following results on strong stabilization of contractive semigroups:

Theorem 2.2.9 [38, 46] *Suppose that for a contractive system* (A, B)

$$H_{cnu}(T) = M_s(T) = M_s(T^*) \ .$$

Then the system is strongly stabilizable by the feedback $-B^*$ *if and only if* $H_u(T)$ *is controllable for* (A, B) *or for* (A^*, B).

Finally, we give the key results on exponential stabilization. As in the case of weak and strong stabilizations, (exact) controllability also plays an important role in exponential stabilization.

Theorem 2.2.10 [71] *If the pair* (A, B) *is exactly controllable, then it is exponentially stabilizable.*

Note that exact controllability is sufficient but not necesary for exponential stabilization.

Exponential stabilization is related to optimization problems involving quadratic performance indices [13, 62]. Techniques used in finite dimensional optimal control, when applied to distributed systems, yield the following necessary and sufficient conditions for exponential stabilization:

Theorem 2.2.11 [3, 56] *For the system* $\dot{x} = Ax + Bu$, $x(0) = x_0$, *the following conditions are equivalent:*

(i) *The system is exponentially stabilizable.*

(ii) *For every initial condition* $x_0 \in H$, *there exists a control* u *such that*

$$\int_0^\infty ([Rx(t), x(t)] + \|u(t)\|^2)dt \ < \infty \ ,$$

where R *is a strictly positive operator.*

(iii) *There exists a nonnegative operator P satisfying the steady state Riccati equation*

$$[PAx, x] + [x, PAx] + [Rx, x] - \|B^*Px\|^2 = 0 \ \ , x \in D(A) \ . \tag{2.7}$$

Moreover, the stabilizing feedback control is $u(t) = -B^*Px(t)$.

Chapter 3

Stability

This chapter is divided into three sections. Each section covers a different approach to characterization of strongly stable (mainly contractive) semigroups. Section 3.1 is an operator theoretic approach and shows that every strongly stable contraction semigroup is related to a shift operator on an L^2 space. In Section 3.2 we develop a new state space decomposition with strong stability in mind. Strongly stable and unstable subspaces are characterized and conditions for strong stability are obtained. Finally, in Section 3.3 we investigate Lyapunov type approaches to strong stability of uniformly bounded semigroups.

3.1 Dilations of Contractions and Strong Stability

Structures of linear operators in finite dimensional spaces are obtained from the theory of Jordan canonical form. Generalization of this theory to infinite dimensional Hilbert space operators is the so called model for an operator. A canonical model

for an operator is a representation of the operator in terms of "simpler" operators which admit "richer" structures. A dilation of an operator is a particular form of a canonical model where the given operator is represented as part (or compression) of a naturally associated operator of a better understood type on a larger space. Thus if U and T are operators on Hilbert spaces H_1 and H_2 , respectively, U is called a dilation of T whenever $H_2 \subseteq H_1$ and $T = P_2U|_{H_2}$, where P_2 is the orthogonal projection onto H_2, and $|_{H_2}$ denotes the restriction to H_2.

The theory of dilations of contractions was initiated by Halmos and was fully developed by Sz.-Nagy and Foias who showed that every contraction operator can be dilated to a unitary operator. Dilations of semigroups were also investigated by these authors.

In this section, strong stability of contraction semigroups will be studied from the point of view of their dilations. Results on dilations can be found in [16, 18, 25, 49, 50, 51]. Here, we will obtain several extensions by studying the problem from the context of strong stability.

Let $T(t)$, $t \geq 0$, be a contractive semigroup with generator A in H. Since $\|T(t)x\| \leq \|x\|$ for all x and $t \geq 0$, $\|T(t)x\|$ is a non-increasing function of t. Next, we have,

$$0 \leq \|T(t)x\|^2 = [T(t)^*T(t)x, x] \leq [x, x] \ ,$$

which shows that the nonnegative contractions $T(t)^*T(t)$ are bounded and non-increasing in t. Hence, they converge strongly to a nonnegative contraction C^2 (say),

$$\lim_{t\to\infty} T(t)^*T(t)x = C^2x \ .$$

Therefore,

$$\lim_{t\to\infty} \|T(t)x\|^2 = \|Cx\|^2.$$

Now, for $x \in D(A)$,

$$\frac{d}{dt}\|T(t)x\|^2 = 2 \text{ Re. } [AT(t)x, T(t)x] \le 0 .$$

Therefore

$$\|x\|^2 - \|T(t)x\|^2 = \int_0^t -2 \text{ Re. } [AT(\sigma)x, T(\sigma)x]d\sigma .$$

Since $\lim_{t\to\infty} \|T(t)x\|^2 < \infty$, we obtain by letting $t \to \infty$:

$$\|x\|^2 = \|Cx\|^2 - \int_0^\infty 2 \text{ Re. } [AT(t)x, T(t)x]dt . \tag{3.1}$$

Let $[x,y]_n$ be the symmetric sesquilinear form on $D(A)$ defined by $[x,y]_n = -[Ax,y] - [x,Ay]$. Then it is nonnegative definite by the fact that A is dissipative. Hence, $[x,y]_n$ defines an inner product on $D(A)$ [8]. Let K be the completion of the subspace $D(A)$ modulo the set of "zero" elements, i.e., $[x,x]_n = 0$. Then (3.1) can be rewritten as

$$\|x\|^2 = \|Cx\|^2 + \int_0^\infty \|T(t)x\|_n^2 dt , \quad \text{for } \; x \in D(A).$$

This shows that for each $x \in D(A)$, $T(t)x$ is in the space $L^2([0,\infty);K)$. Let $\Omega : D(A) \to L^2([0,\infty);K)$ be the linear transformation defined by $(\Omega x)(t) = T(t)x$, then

$$\|x\|^2 = \|Cx\|^2 + \|\Omega x\|_{L^2}^2, \quad \text{for } \; x \in D(A). \tag{3.2}$$

It follows that each x in $D(A)$ can now be represented by the element $Cx \oplus \Omega x$ of the space $\overline{\mathbf{R}(C)} \oplus L^2([0,\infty);K)$, where $\mathbf{R}(\cdot)$ denotes the range. Let $\Sigma : D(A) \to \overline{\mathbf{R}(C)} \oplus L^2([0,\infty);K)$ be defined by

$$\Sigma x = Cx \oplus \Omega x \quad ,$$

then $\|x\| = \|\Sigma x\|$ from (3.2). We therefore have an isometric representation for $D(A)$. Moreover, since $D(A)$ is dense in H, Σ can be extended to all of H. The above shows that the original space H can be "embedded" in the space $\overline{\mathbf{R}(C)} \oplus L^2([0,\infty); K)$.

By considering the representation of $T(t)x$ in the larger space constructed above, we obtain a dilation of $T(t)$. We have

$$\Sigma T(t)x = CT(t)x \oplus \Omega T(t)x \quad , x \in D(A) .$$

Let $L(t)$ be the left shift semigroup on $L^2([0,\infty); K)$ as defined in Example 2.1.1, then we can write

$$(\Omega T(t)x)(\tau) = T(\tau)T(t)x = T(\tau + t)x = (L(t)\Omega x)(\tau).$$

More is true. We can define a second semigroup on $\overline{\mathbf{R}(C)}$ by $V(t)C = CT(t)$. We have,

$$\|V(t)Cx\|^2 = \|CT(t)x\|^2 = \lim_{\tau\to\infty} \|T(\tau)T(t)x\|^2 = \|Cx\|^2.$$

Thus $V(t)$ is also isometric. Therefore,

$$\Sigma T(t)x = V(t)Cx \oplus L(t)\Omega x = (V(t) \oplus L(t))\Sigma x \tag{3.3}$$

for x in $D(A)$, and by continuity, for all x in H.

We conclude that the semigroup

$$U(t) = V(t) \oplus L(t)$$

defined on the space $\overline{\mathbf{R}(C)} \oplus L^2([0,\infty); K)$ is a dilation of the contractive semigroup $T(t)$. From (3.3), since Σ is isometric ($\Sigma^*\Sigma = I$), $T(t)$ admits the representation

$$T(t) = \Sigma^* U(t) \Sigma .$$

Note from (3.3) that the closed subspace $\mathbf{R}(\Sigma)$ is an invariant subspace for $U(t)$. Hence, $T(t)$ is isometrically equivalent to the restriction of $U(t)$ to an invariant subspace.

If $T(t)$ is strongly stable then of course, $C = 0$ and the above representation takes the form

$$T(t) = \Sigma^* L(t)\Sigma \ ,$$

or $T(t)$ is isometrically equivalent to the left shift $L(t)$ on $L^2([0,\infty); K)$, restricted to an invariant subspace.

Suppose now that a contractive semigroup $T(t)$ is such that $T(t) = \Sigma^* L(t)|_N \Sigma$ for some isometry Σ and some invariant subspace N of $L(t)$, then

$$\begin{aligned}
\|T(t)x\| &= \|\Sigma^* L(t)|_N \Sigma x\| \\
&= \|\Sigma^* L(t) P_N \Sigma x\| \\
&\leq \|\Sigma^*\| \, \|L(t) P_N \Sigma x\| \\
&\leq \|L(t) P_N \Sigma x\| \to 0 \text{ as } t \to \infty,
\end{aligned}$$

which shows that $T(t)$ is strongly stable. Thus, we have proved:

Theorem 3.1.1 *A contractive semigroup $T(t)$ is strongly stable if and only if it is isometrically equivalent to the restriction of the left shift semigroup $L(t)$ on $L^2([0,\infty); K)$ to an invariant subspace.*

The above theorem was indicated by Fillmore [18]. One half of it also appears in the context of the theory of scattering [36, 37], where the semigroup related to the scattering operator turns out to be a strongly stable contraction. The shift representation is then a consequence of the application of the translation representation theorem for groups of unitary operators.

We note that in the above theorem the semigroup $T(t)$ can actually be exponentially stable. In other words, an exponentially stable contractive semigroup also has the same shift representation. Although exponential stability implies strong stability, here we will not consider this case; instead we look for weaker conditions for strong stability. The following Proposition provides a means of verifying the lack of exponential stability of contraction semigroups.

Proposition 3.1.1 *If a contractive semigroup $T(t)$ is strongly stable but <u>not</u> exponentially stable, then $\|T(t)\| = 1$ for $t \geq 0$.*

Proof: Since $T(t)$ is a contraction, $\|T(t)\| \leq 1$. Suppose $\|T(t_0)\| < 1$ for some $t_0 > 0$. Then $T(t)$ must be exponentially stable [3]. Therefore, there is no such $t_0 > 0$ and we must have $\|T(t)\| = 1$ for $t \geq 0$. □

Example 3.1.1 Let $\{\phi_n, n = 1, 2, \ldots\}$ be orthonormal basis for the space H and let

$$T(t)x = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{t}{n}} [x, \phi_n]\phi_n \quad .$$

Then $T(t)$ a strongly stable contractive semigroup and

$$\|T(t)\| = \sup_{\|x\|=1} \|T(t)x\| = \lim_{n\to\infty} \mathrm{e}^{-\frac{t}{n}} = 1 \quad .$$

Therefore $T(t)$ is not exponentially stable. △

Note that a contraction semigroup with $\|T(t)\| = 1$, $t \geq 0$, need not be stable in any sense. But, if a strongly stable contractive semigroup is such that $\|T(t)\| = 1$, then it <u>cannot</u> be exponentially stable. A class of semigroups with this property are those contraction semigroups $T(t)$ whose adjoint semigroup $T(t)^*$ are isometries, $\|T(t)^*x\| = \|x\|$ for all x and $t \geq 0$, in other words, $T(t)$ is a co-isometry.

Theorem 3.1.2 [18] *A co-isometric semigroup $T(t)$, $t \geq 0$, is strongly stable if and only if it is unitarily equivalent to the left shift semigroup $L(t)$ on all of $L^2([0,\infty); K)$ where K is an auxiliary Hilbert space.*

Requiring the adjoint semigroup to be isometric is still too strong a condition. We now give a weaker condition for a contractive semigroup to be strongly stable but not exponentially stable.

Theorem 3.1.3 *A contractive semigroup $T(t)$ is strongly stable but not exponentially stable if it is unitarily equivalent to the left shift semigroup on $L^2([0,\infty); K)$ restricted to a reducing subspace.*

Proof: Suppose

$$T(t) = \Sigma^* L(t)|_N \Sigma$$

where $\Sigma : H \rightarrow L^2([0,\infty); K)$ is unitary and N is reducing for $L(t)$. Since N is reducing, it is invariant for both $L(t)$ and $L(t)^*$. Let P_N be the projection onto N. Then,

$$P_N L(t) P_N = L(t) P_N = P_N L(t) = L(t)|_N$$

and

$$P_N L(t)^* P_N = L(t)^* P_N = P_N L(t)^* = L(t)^*|_N \, .$$

Hence, the representation of $T(t)$ can be rewritten as

$$T(t) = \Sigma^* L(t) P_N \Sigma = \Sigma^* P_N L(t) \Sigma \, .$$

We need to show that $\|T(t)x\| \rightarrow 0$ for all x as $t \rightarrow \infty$ and $\|T(t)\| = 1$ for $t \geq 0$. For x in H, we have

$$\|T(t)x\| = \|\Sigma^* L(t) P_N \Sigma x\| \quad \leq \quad \|\Sigma^*\| \, \|L(t) P_N \Sigma x\|$$

$$\leq \|L(t)P_N\Sigma x\| \,.$$

But the left shift semigroup on $L^2([0,\infty);K)$ is strongly stable. Therefore, $\|T(t)x\| \to 0$ as $t \to \infty$, proving the first part of our claim.

To show that $\|T(t)\| = 1$ for $t \geq 0$, first we observe that the adjoint semigroup has the representation

$$T(t)^* = \Sigma^* P_N L(t)^* \Sigma = \Sigma^* L(t)^* P_N \Sigma = \Sigma^* L(t)^*|_N \Sigma \tag{3.4}$$

which shows that $T(t)^*$ is unitarily equivalent to the restriction of $L(t)^*$ to an invariant subspace.

It is well known that the adjoint semigroup $L(t)^*$ of the left shift semigroup is the right shift semigroup $R(t)$, $t \geq 0$, which is an isometry, and the restriction of an isometry to an invariant subspace is again an isometry. Hence, $L(t)^*|_N = L(t)^* P_N$ is isometric and $\|L(t)^* P_N\| = 1$. Since $\Sigma\Sigma^* = \Sigma^*\Sigma = I$ we have $L(t)^* P_N = \Sigma T(t)^* \Sigma^*$ from equation (3.4), and

$$\begin{aligned} 1 = \|L(t)^* P_N\| = \|\Sigma T(t)^* \Sigma^*\| &\leq \|T(t)^*\| \, \|\Sigma\| \, \|\Sigma^*\| \\ &\leq 1 \,. \end{aligned}$$

Therefore, $\|T(t)^*\| = \|T(t)\| = 1$ for $t \geq 0$. This completes the proof. □

We note that a necessary and sufficient condition for strong stability of a contractive semigroup is that the semigroup be isometrically equivalent to the left shift restricted to an invariant subspace, as given in Theorem 3.1.1. On the other hand, Theorem 3.1.2 gives a necessary and sufficient condition for a semigroup to be co-isometric and strongly stable. In this case, the semigroup is unitarily equivalent to the left shift. Our result, then, is the "in between" case where the semigroup is unitarily equivalent to the restriction of the left shift to a reducing subspace, and this

is only sufficient for strong stability. These results may be regarded as outcomes of a certain method of constructing dilations for contractive semigroups as mentioned at the beginning of this section.

There are other methods of constructing dilations for contractive semigroups, however they are not suitable for the study of stability. The theory of dilations for contraction operators (more correctly, discrete semigroups) is well developed [16, 51], which can be applied to semigroups via the notion of a co-generator.

The co-generator of a contractive semigroup $T(t)$ with generator A is defined by

$$T = (A + I)(A - I)^{-1}$$

and it is a contraction. It is easy to see that A is uniquely determined by T. Hence, $T(t)$ is also uniquely determined by T. T^* is the co-generator of the adjoint semigroup $T(t)^*$. It can be shown that any contraction T for which 1 is not an eigenvalue is the co-generator of a contractive semigroup. Properties of a contraction semigroup can be obtained from those of its co-generator and vice versa. We have:

Proposition 3.1.2 [18, 51] *A C_0 semigroup of contractions $T(t)$ is normal, self-adjoint, unitary, or isometric if and only if its co-generator T is normal, self-adjoint, unitary, or isometric, respectively. Moreover, a subspace N is invariant for $T(t)$ if and only if it is invariant for T, in which case $T|_N$ is the co-generator of $T(t)|_N$.*

The following proposition on the relationship between the asymptotic behaviours of T and $T(t)$ is the key to the study of the stability of the semigroup via its co-generator.

Proposition 3.1.3 [51] *Let $T(t)$, $t \geq 0$ be a C_0 semigroup of contractions and let T be its co-generator. Then*

$$\lim_{n\to\infty} \|T^n x\| = \lim_{t\to\infty} \|T(t)x\| \ \ and \ \ \lim_{n\to\infty} \|T^{*n}x\| = \lim_{t\to\infty} \|T(t)^* x\| \ \ for \ \ all \ \ x.$$

It follows that characterizing contractions which are co-generators and whose powers tend to zero strongly is equivalent to characterizing strongly stable contractive semigroups. There are a number of results on the subject, and the pioneering work, which is complete in itself, is that of Sz.-Nagy and Foias [51]. They have classified contractions according to their asymptotic behaviour and characterized each class. Although these results are applicable to strongly stable contractive semigroups, this approach is limited since it is rather difficult to compute the co-generator. But in some cases, given a generator it may be easier to compute the co-generator than the semigroup itself. Then strong stability of the system can be investigated through the application of the abovementioned results.

Example 3.1.2 The right shift semigroup on $L^2[0,\infty)$ defined by

$$(R(t)x)(s) = \begin{cases} 0 & ,0 \leq s < t \\ x(s-t) & ,t \leq s \end{cases}$$

is a semigroup of isometries. Its adjoint is the left shift semigroup $L(t)$:

$$(L(t)x)(s) = x(s+t) \ .$$

We shall show that the co-isometric semigroup $L(t)$ is strongly stable by showing that the powers of its co-generator tend to zero strongly.

The generator $\tilde{A}$ of $R(t)$ is given by

$$\begin{aligned} (\tilde{A}x)(s) &= -\frac{dx}{ds} \\ D(\tilde{A}) &= \{x : x \text{ absolutely continuous}, \frac{dx}{ds} \in L^2([0,\infty); K), x(0) = 0\} \ . \end{aligned}$$

Let

$$Z = (\check{A} + I)(\check{A} - I)^{-1}$$

be the co-generator of $R(t)$. It follows that

$$(Zx)(s) = x(s) - 2\, e^{-s} \int_0^s e^{\sigma} x(\sigma) d\sigma \ ,$$

and it is easy to verify that Z is an isometry. The adjoint of Z

$$(Z^* x)(s) = x(s) - 2\, e^{s} \int_s^{\infty} e^{-\sigma} x(\sigma) d\sigma$$

is the co-generator of $L(t)$.

To show that

$$\|Z^{*k} x\| \to 0 \quad \text{as} \quad k \to \infty \ ,$$

we use a decomposition for the state space that can be summarized as follows [51].

Let Z be an isometry on a Hilbert space H and let $a \in \mathbf{N}(Z^*)$ where $\mathbf{N}(\cdot)$ denotes the nullspace. Then if $(\text{span}\{Z^n a\}_{n=0}^{\infty})^{\perp} = \{0\}$, $\{Z^n a\}_{n=0}^{\infty}$ form an orthogonal basis for H. Letting $f_n = Z^n a$ we have, for $x \in H$,

$$x = \sum_{n=0}^{\infty} [x, f_n] f_n \ .$$

With this decomposition, Z^{*k} acts as a shift of coefficients by k units to the left:

$$Z^{*k} x = \sum_{n=0}^{\infty} [x, f_{n+k}] f_n \ .$$

In our case, we take the isometry Z to be the co-generator of $R(t)$. It turns out that for $a(s) = \sqrt{2}\, e^{-s}$, the functions $Z^n a$, called the Laguerre functions, form an orthonormal basis for $L^2[0, \infty)$.

Hence, letting $f_n(s) = Z^n(\sqrt{2}\, e^{-s})$ the co-generator Z^* of the left shift semigroup has the representation

$$(Z^{*k}x)(s) = \sum_{n=0}^{\infty} [x, f_{n+k}]_{L^2} f_n(s) \quad .$$

Therefore,

$$\begin{aligned} \|(Z^{*k}x)(s)\|_{L^2}^2 &= \sum_{n=0}^{\infty} |[x, f_{n+k}]_{L^2}|^2 \\ &= \sum_{m=k}^{\infty} |[x, f_m]_{L^2}|^2 \quad , \end{aligned}$$

which shows that $\|(Z^{*k}x)(s)\|_{L^2} \to 0$ as $k \to \infty$. △

3.2 Decomposition of the State Space

We develop in this section a decomposition for the Hilbert space H on which a contraction semigroup $T(t)$, $t \geq 0$, acts. This decomposition depends on the strongly stable subspace $M_s(T)$ of the semigroup $T(t)$, defined by:

$$M_s(T) = \{x \in H : T(t)x \to 0 \;\text{ as }\; t \to \infty\}.$$

Consequently, conditions for stability can be found by requiring this subspace to be all of H.

We must note that decomposing a state space into stable and unstable subspaces is a familiar technique in the study of finite dimensional systems. This technique has also been used in the case of weak stability of contraction semigroups [5], and in exponential stabilizability [66].

Let A be the generator of a contraction semigroup $T(t)$ and define the set

$$\mathcal{N} = \{x \in D(A) : \text{ Re. } [Ax, x] = 0\}.$$

Then, of course, $\mathcal{N}$ is a subset of $D(A)$. Moreover,

Proposition 3.2.1 *$\mathcal{N}$ is a closed subspace.*

Proof: Let

$$S = (A + I)(A - I)^{-1}$$

be the co-generator of the semigroup $T(t)$. Then S is a contraction which does not admit 1 as its eigenvalue, and $A = (S + I)(S - I)^{-1}$. Define the set

$$H_1 = \{y \in H : \|Sy\| = \|y\|\} ,$$

then:

$$\begin{aligned} 0 &= \|Sy\|^2 - \|y\|^2 \\ &= [Sy, Sy] - [y, y] - \text{ Re. } [Sy, y] + \text{ Re. } [Sy, y] \\ &= \text{ Re. } [(S + I)y, (S - I)y] . \end{aligned}$$

Let $(S - I)y = x$, then $x \in D(A)$ and $y = (S - I)^{-1}x$. Therefore,

$$\begin{aligned} 0 &= \text{ Re. } [(S + I)(S - I)^{-1}x, x] \\ &= \text{ Re. } [Ax, x] , \end{aligned}$$

or $H_1 \subseteq \mathcal{N}$.

Now, suppose $x \in \mathcal{N}$, then $0 = \text{ Re. } [Ax, x] = \text{ Re. } [(S + I)(S - I)^{-1}x, x]$. Let $(S - I)^{-1}x = y$ so that $x = (S - I)y$. Hence, $0 = \|Sy\|^2 - \|y\|^2$, or $\|Sy\| = \|y\|$. We therefore conclude that:

$$\begin{aligned} \mathcal{N} &= \{x \in D(A) : \text{ Re. } [Ax, x] = 0\} \\ &= \{y \in H : \|Sy\| = \|y\|\} \\ &= H_1 . \end{aligned}$$

But, since S is a contraction, $\|Sy\| = \|y\| \iff S^*Sy = y$, or $(I - S^*S)y = 0$. This shows that

$$H_1 = \mathbf{N}(I - S^*S) = \mathcal{N} ,$$

where $\mathbf{N}(\cdot)$ denotes the nullspace. Hence, $\mathcal{N}$ is a closed subspace as expected. $\square$

Now, recall that the isometric subspace of a contractive semigroup $T(t)$ is defined as

$$H_i(T) = \{x \in H : \|T(t)x\| = \|x\| \text{ for } t \geq 0\} .$$

Since $T(t)$ is contractive, the operators $(I - T(t)^*T(t))$, $t \geq 0$, are nonnegative, and clearly

$$H_i(T) = \bigcap_{t\geq 0} \mathbf{N}(I - T(t)^*T(t)) .$$

The sets $\mathbf{N}(I - T(t)^*T(t))$ are decreasing in t since for $x \in \mathbf{N}(I - T(t+s)^*T(t+s))$, $t, s \geq 0$, we have

$$\begin{aligned} \|x\|^2 = \|T(t+s)x\|^2 &= \|T(s)T(t)x\|^2 \\ &\leq \|T(t)x\|^2 \leq \|x\|^2 \end{aligned}$$

by the fact that $T(t)$, $t \geq 0$, is a contraction. Therefore, $\|T(t)x\| = \|x\|$ or $x \in \mathbf{N}(I - T(t)^*T(t))$, proving that for $t_2 \geq t_1$,

$$\mathbf{N}(I - T(t_2)^*T(t_2)) \subseteq \mathbf{N}(I - T(t_1)^*T(t_1)) .$$

It then follows that,

$$\begin{aligned} H_i(T) &= \bigcap_{t\geq 0} \mathbf{N}(I - T(t)^*T(t)) \\ &= \{x \in H : \lim_{t\to\infty} \|T(t)x\| = \|x\|\}. \end{aligned} \tag{3.5}$$

We have seen in the proof of Proposition 3.2.1 that $\mathcal{N} = H_1 = \{x \in H : \|Sx\| = \|x\|\}$. We now define the subspaces

$$H_n = \{x \in H : \|S^n x\| = \|x\|\}, \quad \text{for } n \geq 0 .$$

Then $H_{n+1} \subseteq H_n$. To see this we only have to note that

$$\|x\| = \|S^{n+1}x\| \leq \|S^n x\| \leq \|x\|, \quad \text{for } x \in H_{n+1}.$$

Hence, $x \in H_n$. Thus, the subspace

$$H_i(S) = \{x \in H : \|S^n x\| = \|x\| \text{ for } n = 1, 2, \ldots\} \ ,$$

which is the isometric subspace of the co-generator S, can also be expressed as

$$\begin{aligned} H_i(S) &= \bigcap_{n \geq 0} H_n \\ &= \{x \in H : \lim_{n \to \infty} \|S^n x\| = \|x\|\} . \end{aligned} \tag{3.6}$$

This shows that $H_i(S) \subseteq H_1 = \mathcal{N}$. But, since $\lim_{n\to\infty} \|S^n x\| = \lim_{t\to\infty} \|T(t)x\|$ for all x, we have actually shown that (3.5) and (3.6) are equivalent. More is true as is shown in the next proposition.

Proposition 3.2.2 *The subspace $H_i(T)$ is the largest subspace of $\mathcal{N}$ which is invariant for $T(t)$.*

Proof: Suppose $\mathcal{M} \subset \mathcal{N}$ is an invariant subspace of $T(t)$. Then, for $x \in \mathcal{M}$, $T(t)x$ is also in $\mathcal{M}$ and

$$0 = \text{ Re. } [AT(t)x, T(t)x] = \frac{1}{2} \frac{d}{dt} \|T(t)x\|^2.$$

Therefore, $\|T(t)x\| = \|x\|$, or $x \in H_i(T)$. $\square$

Let $\mathcal{N}^\perp$ be the orthogonal complement in H of $\mathcal{N}$, then we have

$$\begin{aligned} H &= \mathcal{N} \oplus \mathcal{N}^\perp \\ &= H_i(T) \oplus L \oplus \mathcal{N}^\perp \end{aligned} \tag{3.7}$$

where L is the orthogonal complement of $H_i(T)$ in $\mathcal{N}$. We can now prove:

Theorem 3.2.1 *If the co-generator S of a contractive semigroup $T(t)$ is nonnegative, then the semigroup is strongly stable.*

Proof: We have shown that $\mathcal{N} = H_1 = \mathbf{N}(I - S^*S)$. Let $x \in \mathcal{N}$, then $S^*Sx = x = S^2x$ since S is self-adjoint. Set $Sx = y$ so that $Sy = S^2x = x$. Therefore, $S(x-y) = -(x-y)$. But since S is nonnegative we must have $x = y$. Hence, if $x \in \mathcal{N}$ then $Sx = x$, which implies that 1 is an eigenvalue of S. This is not possible by the fact that S is a co-generator. Therefore $Sx = x \Rightarrow x = 0$, or $\mathcal{N} = \{0\}$, which further implies that $H_i(T) = \{0\}$, i.e., $T(t)$ is c.n.i. . Hence, $T(t)$ is also c.n.u., therefore it is weakly stable [5]. Moreover, since S is self-adjoint, so is $T(t)$. In this case weak stability implies strong stability. □

Remark: We note that the generator of a contractive semigroup which satisfies the conditions of the above theorem also has some special properties. First, A is self-adjoint since $T(t)$ is. Also, we have for all x, $0 \le [Sx, x] = [(A+I)(A-I)^{-1}x, x]$. Letting $(A-I)^{-1}x = y$,

$$\begin{aligned} 0 &\le [(A+I)y, (A-I)y] \\ &= \|Ax\|^2 - \|x\|^2 - 2i \text{ Im. } [Ax, x] \\ &= \|Ax\|^2 - \|x\|^2 \end{aligned}$$

since $[Ax, x]$ is real. This shows that $\|x\|^2 \leq \|Ax\|^2$ for all $x \in D(A)$ or A is invertible. Also, since A is dissipative and self-adjoint, $-A \geq 0$ and it is also invertible. $-A^{-1}$ is called an abstract potential operator [70]. $\triangle$

For a contraction semigroup $T(t)$, $t \geq 0$, the operator C defined by

$$C^2 = \lim_{t \to \infty} T(t)^* T(t)$$

which was introduced in Section 3.1 is a nonnegative contraction. It follows from Proposition 3.1.3 that

$$\|Cx\|^2 = \lim_{n \to \infty} \|S^n x\|^2$$

where S is the co-generator of $T(t)$. We can therefore define the isometric subspace $H_i(T)$ as

$$H_i(T) = \{x \in H : \|Cx\| = \|x\|\}.$$

But, since C is a nonnegative contraction we have

$$\begin{aligned} H_i(T) &= \{x \in H : Cx = x\} \\ &= \mathbf{N}(I - C) . \end{aligned}$$

This shows that $H_{cni}(T) = \overline{\mathbf{R}(I - C)}$. The decomposition (3.7) now becomes

$$H = \mathbf{N}(I - C) \oplus L \oplus \mathcal{N}^{\perp} . \tag{3.8}$$

Therefore, $L \oplus \mathcal{N}^{\perp} = \mathbf{N}(I - C)^{\perp} = \overline{\mathbf{R}(I - C)}$. We then have, since C is self-adjoint,

$$H = \mathbf{N}(I - C) \oplus \overline{\mathbf{R}(I - C)} . \tag{3.9}$$

Next, we further decompose $\overline{\mathbf{R}(I - C)}$. First, we observe that $\mathbf{N}(I - C) \perp \mathbf{N}(C)$ since, for x in $\mathbf{N}(I - C)$ and y in $\mathbf{N}(C)$:

$$[x, y] = [Cx, y] = [x, Cy] = 0 .$$

Therefore, by (3.9)

$$\mathbf{N}(C) \subseteq \mathbf{N}(I-C)^{\perp} = \overline{\mathbf{R}(I-C)}$$

and we have

$$\begin{aligned} H &= \mathbf{N}(I-C) \oplus M \oplus \mathbf{N}(C) \qquad (3.10) \\ &= \overline{\mathbf{R}(C)} \oplus \mathbf{N}(C) \end{aligned}$$

where M is the orthogonal complement of $\mathbf{N}(C)$ in $\overline{\mathbf{R}(I-C)}$. It follows from (3.9) and (3.10) that

$$M = \overline{\mathbf{R}(C)} \bigcap \overline{\mathbf{R}(I-C)} \, . \qquad (3.11)$$

Hence, the decomposition (3.10) becomes

$$H = \mathbf{N}(I-C) \oplus (\overline{\mathbf{R}(C)} \bigcap \overline{\mathbf{R}(I-C)}) \oplus \mathbf{N}(C) \, . \qquad (3.12)$$

Now,

$$\begin{aligned} \mathbf{N}(C) &= \{x \in H : Cx = 0\} \\ &= \{x \in H : \lim_{t\to\infty} \|T(t)x\| = 0\} \\ &= M_s(T) \, . \end{aligned}$$

Therefore,

Theorem 3.2.2 *Let $T(t)$ be a contractive semigroup on H. Then H admits the orthogonal decomposition*

$$\begin{aligned} H &= H_i(T) \oplus M \oplus M_s(T) \\ &= \mathbf{N}(I-C) \oplus M \oplus M_s(T) \qquad (3.13) \end{aligned}$$

where

$$\begin{aligned}
C^2 &= \lim_{t\to\infty} T(t)^* T(t) \\
M &= \overline{\mathbf{R}(C)} \bigcap \overline{\mathbf{R}(I-C)} \\
M_s(T) &= \{x \in H : \lim_{t\to\infty} \|T(t)x\| = 0\} \, .
\end{aligned}$$

Remark: We note the following properties of each subspace of the decomposition:

(i) For all x in $\mathbf{N}(I-C)$, $Cx = x$. Therefore, $C = C^2$ on this subspace and it is invariant for C. It is also invariant for $T(t)$, since $\mathbf{N}(I-C) = H_i(T)$ and $T(t)$ is isometric on $\mathbf{N}(I-C)$.

(ii) $M_s(T)$ is invariant for $T(t)$ since

$$T(t)x \to 0 \;\text{ as }\; t \to \infty$$

on this subspace. Clearly, it is also invariant for C.

(iii) For $x \in M$, $\|Cx\| < \|x\|$. To see this, we observe that if $\|Cx\| = \|x\|$ then $x \in \mathbf{N}(I-C) = H_i(T)$, which is not possible. Therefore, we conclude that $\|Cx\| < \|x\|$ by the fact that C is a contraction. In addition, since $H_i(T)$ and $M_s(T)$ are both invariant for $T(t)$, M is invariant for $T(t)^*$. $\triangle$

We now investigate the stability of contraction semigroups using the decomposition developed above. For a contraction semigroup to be strongly stable, $M_s(T)$ must be all of H. That is, $\mathbf{N}(I-C)$ and M both must be trivial. The subspace $\mathbf{N}(I-C)$ is the isometric subspace of $T(t)$ as we have seen above. Therefore, a strongly stable contractive semigroup must necessarily be completely non-isometric.

Proposition 3.2.3 *Let $T(t)$ be a contractive semigroup with generator A on H. Let P be the linear bounded nonnegative operator defined by*

$$[Px, x] = \int_0^\infty -2 \text{ Re. } [AT(t)x, T(t)x] dt \;, x \in D(A). \tag{3.14}$$

Then, $T(t)$ is c.n.i. if and only if $P > 0$.

Proof: We have, for $x \in D(A)$,

$$\|x\|^2 - \|Cx\|^2 = \int_0^\infty -2 \text{ Re. } [AT(t)x, T(t)x] dt \;. \tag{3.15}$$

Set

$$[\Pi x, x] = \int_0^\infty -2 \text{ Re. } [AT(t)x, T(t)x] dt$$

where $\Pi = I - C^2 \geq 0$. This shows that if there exists a P satisfying (3.14), then $P = \Pi$.

By definition, $Px = \lim_{t\to\infty}(I - T(t)^*T(t))x$, $x \in D(A)$. Since $D(A)$ is dense, P can be extended to all of H. We have

$$\mathbf{N}(C) = \lim_{t\to\infty} \mathbf{N}(I - T(t)^*T(t)) = H_i(T) \;.$$

Therefore, $\mathbf{N}(P) = H_i(T)$, which proves the proposition. □

An immediate consequence of Proposition 3.2.3 is the following stability result.

Corollary 3.2.1 *If there exists a linear bounded positive operator P satisfying (3.14), then $T(t)$ is weakly stable.*

We can actually state a necessary and sufficient condition for strong stability along the same lines as Proposition 3.2.3.

Theorem 3.2.3 *A contraction semigroup $T(t)$ with generator A is strongly stable if and only if*

$$\|x\|^2 = \int_0^\infty -2\, Re.\, [AT(t)x, T(t)x]dt \quad , x \in D(A)\,. \tag{3.16}$$

Proof: Suppose $T(t)$ is strongly stable. Then $Cx = 0$ in (3.15) and consequently (3.16) holds. Conversely, suppose (3.16) holds, then replacing x by $T(t)x$, we have

$$\|T(t)x\|^2 = \int_t^\infty -2\ \mathrm{Re.}\, [AT(\sigma)x, T(\sigma)x]d\sigma$$

for $x \in D(A)$. Letting $t \to \infty$, we see that is $T(t)$ strongly stable on $D(A)$. $T(t)$ being a contraction and $D(A)$ being dense in H, this extends to all of H. □

If the generator A has compact resolvent, weak stability implies strong stability for any C_0 semigroup. In the case of contractions, the compact resolvent assumption gives us more as in the next theorem.

Theorem 3.2.4 [39, 46] *Let $T(t)$ be a contraction semigroup with generator A. If A has compact resolvent, then $T(t)$ is strongly stable if and only if it is c.n.i.*

To proceed further, we need the following result due to Fillmore and Williams:

Lemma 3.2.1 [19] *Let A and B be bounded nonnegative operators on H. Define $A : B = A(A+B)^{-1}B$. Then $A : B$ is also bounded and nonnegative and*

$$\mathbf{R}(A) \bigcap \mathbf{R}(B) \subset \mathbf{R}(A : B)\,.$$

Taking $A = C$ and $B = (I - C)$, we find $\mathbf{R}(C) \bigcap \mathbf{R}(I - C) \subset \mathbf{R}(C - C^2)$.

One can actually obtain this directly as follows. Let $y \in \mathbf{R}(C) \bigcap \mathbf{R}(I - C)$, then $y = Cx = (I - C)z$ for some x and z in H. Therefore $Cx = z - Cz$, or $C(x + z) = z$,

which shows that $z \in \mathbf{R}(C)$. We then must have $z = Cw$ for some w. Substituting this, we find $y = (C - C^2)w$, proving the above result.

More can be shown. Let $x \in \mathbf{R}(C - C^2)$, then $x = (C - C^2)v = (I - C)Cv = C(I - C)v$ for some v. Hence, $\mathbf{R}(C - C^2) \subset \mathbf{R}(C) \bigcap \mathbf{R}(I - C)$ and we conclude:

$$\mathbf{R}(C) \bigcap \mathbf{R}(I - C) = \mathbf{R}(C - C^2)$$

or, $\overline{\mathbf{R}(C - C^2)} \subseteq M$.

The next step is to require the subspace M to be trivial. We have:

Lemma 3.2.2 *The subspace M in (3.13) is trivial if and only if C is a projection operator ($C = C^2$).*

Proof: $\Rightarrow$ Since $\overline{\mathbf{R}(C - C^2)} \subseteq M$, if $M = \{0\}$ then $\mathbf{R}(C - C^2) = \{0\}$ as well. This implies that $C - C^2 = 0$ or, since C is already nonnegative, C is a projection.

$\Leftarrow$ If $C (= C^2)$ is a projection then

$$\begin{aligned} \mathbf{N}(C) &= \mathbf{R}(C)^{\perp} = \overline{\mathbf{R}(I - C)} \\ \mathbf{N}(I - C) &= \mathbf{R}(I - C)^{\perp} = \overline{\mathbf{R}(C)} \quad \text{and} \\ H &= \mathbf{N}(I - C) \oplus \mathbf{N}(C) \ , \end{aligned}$$

which shows that $M = \{0\}$. This completes the proof. □

We then have:

Theorem 3.2.5 [1] *A c.n.i. contraction semigroup $T(t), t \geq 0$, over H is strongly stable if and only if C is an orthogonal projection on H.*

The problem is then to find conditions for C to be a projection operator.

[1]Theorem 3.2.5 was proved by a different method by Levan and Rigby [46].

Theorem 3.2.6 *For a contraction semigroup $T(t)$, the operator C is a projection under any of the following conditions:*

(i) *The generator A has compact resolvent.*

(ii) *$T(t)$ is normal for $t \geq 0$.*

(iii) *$T(t)$ is self-adjoint.*

(iv) *$T(t)$ is co-isometric.*

Proof:

(i) If A has compact resolvent weak and strong stabilities are equivalent, or $M_s(T) = W(T)$. Hence,

$$M_s(T)^{\perp} = W(T)^{\perp} \subseteq H_u(T) \subseteq H_i(T) \; .$$

But this implies $H_i(T) \oplus M \subseteq H_i(T)$ by the decomposition (3.13). Therefore we must have $C = C'^2$.

(ii)-(iii) First, we note that a self-adjoint semigroup is also normal since $T(t)^*T(t) = T(t)T(t) = T(t)T(t)^*$. We have, from the spectral theory of normal operators [58], $H_{cnu}(T) = M_s(T)$. Therefore,

$$\begin{aligned} M_s(T) &\subseteq M_s(T) \oplus M = H_{cni}(T) \\ &\subseteq H_{cnu}(T) = M_s(T) \; . \end{aligned}$$

Hence, $M = \{0\}$ and C is a projection.

(iv) Suppose $T(t)^*$ is isometric, then $T(t)T(t)^* = I$. So, the self-adjoint operators $T(t)^*T(t)$ are such that

$$(T(t)^*T(t))\,(T(t)^*T(t)) = T(t)^*T(t)$$

or, they are projections. Therefore

$$C^2 = \lim_{t \to \infty} T(t)^* T(t) = C \ .$$

□

In the previous section we presented a characterization of strongly stable co-isometric semigroups via the dilation theory approach (Theorem 3.1.2). We can now give another characterization via the decomposition developed above.

Lemma 3.2.3 *A co-isometric semigroup is strongly stable if and only if it is c.n.i..*

Proof: Suppose $T(t)$, $t \geq 0$ is co-isometric. Then $C = C^2$ from Theorem 3.2.6 and $M_s(T) = H_i(T)^\perp = H_{cni}(T)$ from (3.13). If $T(t)$ is c.n.i. then $H = H_{cni}(T) = M_s(T)$, hence $T(t)$ is strongly stable.

On the other hand, if $T(t)$ is strongly stable, then it is c.n.i. . Otherwise, there would exist an x for which $\|T(t)x\| = \|x\|$ for $t \geq 0$, which contradicts the assumption. □

Example 3.2.1 Let $\{\phi_n, n = 1, 2, \ldots\}$ and $\{\psi_m, m = 1, 2, \ldots\}$ be biorthonormal basis for the Hilbert space H. That is,

$$\begin{aligned} [\phi_n, \phi_m] = [\psi_n, \psi_m] &= 0 \ \text{ if } n \neq m \ , \\ \|\phi_n\| = \|\psi_n\| &= 1 \ \text{ for all } n \end{aligned}$$

and

$$[\phi_n, \psi_m] = 0 \ \text{ for all } n, m \ .$$

Let $T(t)$, $t \geq 0$, be given by

$$T(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n + i\beta_n)t} [x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{i\gamma_m t} [x, \psi_m]\psi_m$$

where α_n, β_n and γ_m are all real numbers and $\alpha_n > 0$. Then, for all $x \in H$

$$\begin{aligned} \|T(t)x\|^2 &= \sum_{n=1}^{\infty} e^{-2\alpha_n t} |[x, \phi_n]|^2 + \sum_{m=1}^{\infty} |[x, \psi_m]|^2 \\ &\leq \sum_{n=1}^{\infty} |[x, \phi_n]|^2 + \sum_{m=1}^{\infty} |[x, \psi_m]|^2 \\ &= \|x\|^2 \end{aligned}$$

which shows that $T(t)$ is a contractive semigroup. Hence, we can apply Theorem 3.2.2 to decompose the space H. First we have

$$\begin{aligned} C^2 x &= \lim_{t \to \infty} T(t)^* T(t)x \\ &= \sum_{m=1}^{\infty} [x, \psi_m]\psi_m \ . \end{aligned}$$

Therefore

$$C^2 x = P_\psi x$$

where P_ψ is the orthogonal projection onto the subspace spanned by $\{\psi_m, m = 1, 2, \ldots\}$. Then C is a projection and

$$M = \{0\} \ .$$

In fact, since

$$T(t)^* x = \sum_{n=1}^{\infty} e^{(-\alpha_n - i\beta_n)t} [x, \phi_n]\phi_n + \sum_{m=1}^{\infty} e^{-i\gamma_m t} [x, \psi_m]\psi_m$$

we have $\|T(t)x\| = \|T(t)^* x\|$, which shows that $T(t)$ is normal. According to Theorem 3.2.6 the operator C has to be a projection in this case.

Now, it is easy to see that

$$\begin{aligned} \mathbf{N}(I - C) = H_i(T) &= H_u(T) \\ &= \overline{\text{span}\{\psi_m\}_{m=1}^{\infty}} \end{aligned}$$

and

$$
\begin{aligned}
M_s(T) = \mathbf{N}(C) = H_{cni}(T) &= H_{cnu}(T) \\
&= \overline{\operatorname{span}\{\phi_n\}_{n=1}^{\infty}} \ .
\end{aligned}
$$

Hence,

$$
\begin{aligned}
H &= H_i(T) \oplus M_s(T) \\
&= \overline{\operatorname{span}\{\psi_m\}_{m=1}^{\infty}} \oplus \overline{\operatorname{span}\{\phi_n\}_{n=1}^{\infty}}
\end{aligned}
$$

and $T(t)$ is strongly stable if and only if $\psi_m = 0$ for $m = 1, 2, \ldots$. $\triangle$

Example 3.2.2 Consider the heat equation on a bounded domain:

$$
\frac{\partial x(t,\xi)}{\partial t} = \frac{\partial^2 x(t,\xi)}{\partial \xi^2} \quad , \xi \in [0, 2\pi] \, , t \geq 0
$$

with the boundary conditions

$$
\begin{aligned}
x(t,0) &= x(t,2\pi) \\
x_\xi(t,0) &= x_\xi(t,2\pi) \ .
\end{aligned}
$$

Let $H = L^2[0, 2\pi]$ and let $A = \frac{\partial^2}{\partial \xi^2}$ with

$$
\begin{aligned}
D(A) = \{&x \in H : x \text{ and } x' \text{ absolutely continuous, } x' \text{ and } x'' \in H, \\
&x(0) = x(2\pi), \ x'(0) = x'(2\pi)\} \ .
\end{aligned}
$$

Then A is self-adjoint and dissipative and generates a compact contraction semigroup $T(t)$, $t \geq 0$, given by

$$
T(t)x = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2 t} \, [x, \phi_n]_{L^2} \ \phi_n
$$

where

$$\phi_n(\xi) = \frac{e^{-in\xi}}{\sqrt{2\pi}}$$

and $\{\phi_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2[0, 2\pi]$ [3].

Since A is self-adjoint so is $T(t)$ and according to Theorem 3.2.6 the operator C is a projection. Then, we have $H = \mathbf{N}(I - C) \oplus M_s(T)$, where

$$\mathbf{N}(I - C) = H_i(T) = H_u(T) = \text{span}\{\phi_0\}$$

and

$$M_s(T) = H_{cnu}(T) = \overline{\text{span}\{\phi_n, n = \pm 1, \pm 2, \ldots\}} \ .$$

Therefore, $T(t)$ is not stable in any sense. $\triangle$

3.3 Extensions of Lyapunov Equation

Let A be the generator of a C_0 semigroup $T(t)$, $t \geq 0$, over a Hilbert space H. A necessary and sufficient condition for the semigroup to be exponentially stable is the existence of a positive solution P to the Lyapunov equation

$$[PAx, x] + [x, PAx] = -\|x\|^2, \ \ x \in D(A) \ . \tag{3.17}$$

In this section, we wish to generalize (3.17) to other types of stabilities. We begin by pointing out an interesting fact concerning Pazy's result on exponential stability (Theorem 2.2.7), which, as far as we know, has not been pointed out before.

Let $T(t)$, $t \geq 0$, be an exponentially stable C_0 semigroup. Then, replacing x by $T(t)x$ in (3.17) and integrating both sides, we obtain

$$-\int_0^t \|T(\sigma)x\|^2 d\sigma = [PT(t)x, T(t)x] - [Px, x] < 0 \ .$$

Therefore,

$$[PT(t)x, T(t)x] < [Px, x] \tag{3.18}$$

for every x in $D(A)$. Since the semigroup is exponentially stable, it is uniformly bounded, and since $D(A)$ is dense, (3.18) holds for all x in H.

It is easy to see that the operator P satisfying (3.17) is given by

$$[Px, x] = \int_0^\infty \|T(t)x\|^2 dt \ .$$

Thus, one can use P to define a new norm which is equivalent to the original norm if the conditions of Pazy's theorem are satisfied. Denote by Q the square root of P, $P = Q^2$. Then (3.18) can be rewritten as

$$\|QT(t)Q^{-1}y\| < \|y\|^2 \quad \text{for all} \ \ y \in H,$$

where we set $Qx = y$. Therefore it is evident that Pazy's results are necessary and sufficient conditions for an exponentially stable semigroup to be similar to a strictly contractive one.

Now, let $\|\cdot\|_n$ be a new norm on H or, depending on the case, just on a dense subspace of H, and consider the equation

$$[PAx, x] + [x, PAx] = -\|x\|_n^2 \ , \quad x \in D(A) \ . \tag{3.19}$$

If the two norms $\|\cdot\|$ and $\|\cdot\|_n$ are equivalent, i.e.,

$$k_1\|x\| \leq \|x\|_n \leq k_2\|x\| \quad , x \in H$$

for some $k_1 > 0$ and $k_2 > 0$, then of course, Datko's result still holds and the semigroup is exponentially stable.

This suggests that one should study equation (3.19) with

(i) $\|x\|_n \le k_2\|x\|$,

and

(ii) $k_1\|x\| \le \|x\|_n$.

These two extensions of the Lyapunov equation are the main subjects of our study in this section. We first prove a general existence theorem.

Theorem 3.3.1 *A necessary and sufficient condition for the existence of a linear bounded operator $P \ge 0$ satisfying*

$$2 \text{ Re. } [PAx, x] = -\|x\|_n^2 , \quad x \in D(A) , \tag{3.20}$$

where $\|\cdot\|_n$ is a new norm on $D(A)$ (or H), is the convergence of the integral

$$\int_0^\infty \|T(t)x\|_n^2 dt < \infty \quad \textit{for } x \in D(A).$$

Proof: Suppose there exists a linear bounded nonnegative solution P. Then, replacing x by $T(t)x$ and integrating (3.20), we have for $x \in D(A)$

$$[PT(t)x, T(t)x] - [Px, x] = -\int_0^t \|T(\sigma)x\|_n^2 d\sigma$$

or

$$[PT(t)x, T(t)x] = [Px, x] - \int_0^t \|T(\sigma)x\|_n^2 d\sigma \ge 0 .$$

Hence, we have

$$\int_0^t \|T(\sigma)x\|_n^2 d\sigma \le [Px, x] < \infty \tag{3.21}$$

for every x in $D(A)$, and taking the limit as $t \to \infty$, we conclude that the infinite integral converges.

Conversely, suppose

$$\int_0^\infty \|T(t)x\|_n^2 dt < \infty \quad \text{for } x \in D(A) .$$

Define P on $D(A)$ by

$$[Px, x] = \int_0^\infty \|T(t)x\|_n^2 dt .$$

Then P is linear, bounded, self-adjoint, and positive on $D(A)$. Since $D(A)$ is dense, we can extend $P \geq 0$ to all of H.

Consider, for $x \in D(A)$

$$[PT(t)x, T(t)x] = \int_t^\infty \|T(\sigma)x\|_n^2 d\sigma .$$

Differentiating with respect to t yields

$$[PAT(t)x, T(t)x] + [T(t)x, PAT(t)x] = -\|T(t)x\|_n^2 .$$

Setting $t = 0$, we have the desired relation. □

Note that the proof remains valid even if $\|\cdot\|_n$ is a seminorm, i.e., $\|x\|_n = 0$ does not necessarily imply that $x = 0$.

We now investigate the Lyapunov equation

$$[PAx, x] + [x, PAx] = -\|x\|_n^2 \quad , x \in D(A). \tag{3.22}$$

where $\|\cdot\|_n$ is a new norm on H and is weaker than the original norm, i.e., for some $k_2 > 0$:

$$\|x\|_n \leq k_2\|x\|, \quad x \in H.$$

Suppose that there exists a nonnegative, bounded solution P of (3.22). Then, we know from the proof of Theorem 3.3.1 that P can be expressed as

$$[Px, x] = \int_0^\infty \|T(t)x\|_n^2 dt \tag{3.23}$$

for x in $D(A)$.

However, in this case, the integral in (3.23) converges for all x in H. To prove this, we note that

$$\begin{aligned}\|T(t)(x_i - x)\|_n &\leq k_2\|T(t)(x_i - x)\| \\ &\leq k_2\|T(t)\|\ \|x_i - x\| \\ &\leq k_2 M e^{\omega t}\|x_i - x\|.\end{aligned}$$

If $\|x_i - x\| \to 0$, then $\|T(t)x_i - T(t)x\|_n \to 0$ uniformly on compact intervals of $[0, \infty)$. Hence, the inequality (3.21) holds for all x since $D(A)$ is dense in H. This means that the expression for P given by (3.23) is valid not only on $D(A)$ but on all of H.

Now, consider for x in H

$$[PT(t)x, T(t)x] = \int_t^\infty \|T(\sigma)x\|_n^2 d\sigma\ .$$

Then, we have

$$\lim_{t\to\infty}[PT(t)x, T(t)x] = 0, \quad x \in H. \tag{3.24}$$

In addition, since $\|\cdot\|_n$ is a new norm, P is positive, i.e., $[Px, x] > 0$ for all x. Therefore, if we further require $T(t)$, $t \geq 0$, to be uniformly bounded, $\|T(t)\| \leq M$, then (3.24) implies that $T(t)$ is weakly stable. We summarize the result below:

Theorem 3.3.2 *Let $T(t)$, $t \geq 0$, be uniformly bounded with generator A. If there exists a linear bounded positive solution of the Lyapunov equation*

$$[PAx, x] + [x, PAx] = -\|x\|_n^2\ , \quad x \in D(A)$$

where $\|\cdot\|_n$ is a new norm on H and for some $k_2 > 0$:

$$\|x\|_n \leq k_2\|x\| \tag{3.25}$$

then, $T(t)$ is weakly stable.

We note that if (3.25) holds for a new norm, then it admits the representation

$$\|x\|_n^2 = [Vx, x]$$

for some linear, bounded, self-adjoint operator V.

Recently, Miyaji investigated Lyapunov type approaches to stability of C_0 semigroups. He proved the following theorem:

Theorem 3.3.3 [48] *Let $T(t)$, $t \geq 0$, be a C_0 semigroup. If there exists a linear bounded operator B on H and an $\alpha > 0$ such that for every x in H*

$$\alpha\|x\|^2 \leq \int_0^\infty \|B^*T(t)x\|^2 dt < \infty \,, \tag{3.26}$$

then, $T(t)$ is strongly stable.

This result is an application of the first type of extension of the Lyapunov equation. Because, if we let $\|x\|_n^2 = [BB^*x, x]$, then we can define a nonnegative operator P by

$$[Px, x] = \int_0^\infty \|B^*T(t)x\|^2 dt$$

and P satisfies

$$[PAx, x] + [x, PAx] = -\|B^*x\|^2, \quad x \in D(A).$$

Also, since B is bounded, $\|x\|_n \leq \|B^*\|\|x\|$, and in which case, we have shown that

$$[PT(t)x, T(t)x] \to 0 \quad \text{as} \quad t \to \infty, \qquad \text{for all} \quad x \in H. \tag{3.27}$$

Hence, if there exists an $\alpha > 0$ such that (3.26) holds, then P defines an equivalent norm, and we conclude from (3.27) that $T(t)$ is strongly stable.

If P is positive then (3.27) implies that the uniformly bounded semigroup $T(t)$ is weakly stable. A condition for P to be positive is that the pair (A^*, B) be controllable. We also note that if $T(t)$ is uniformly bounded and $\mathbf{N}(B^*)$ is trivial, then all conditions of Theorem 3.3.2 are satisfied for weak stability.

As a final comment on Theorem 3.3.3, we will show that as in Pazy's theorem, it also results in a sufficient condition for a C_0 semigroup to be similar to a contraction. Since

$$[PT(t)x, T(t)x] = \int_t^\infty \|B^*T(\sigma)x\|^2 d\sigma,$$

it follows that for x in H,

$$[PT(t)x, T(t)x] < [Px, x].$$

Since P is bounded from below, letting $P = Q^2$, we conclude that $T(t)$ is similar to a contraction semigroup $C(t)$ (say) given by

$$C(t) = QT(t)Q^{-1}.$$

Since $T(t)$ is strongly stable, so is $C(t)$. The generator of $C(t)$ is $A_C = QAQ^{-1}$ with $D(A_C) = Q^{-1}D(A)$. Rewriting the corresponding Lyapunov equation as

$$[QAQ^{-1}y, y] + [y, QAQ^{-1}y] = -\|B^*Q^{-1}y\|^2$$

where $Qx = y$, we see that A_C satisfies

$$[A_C y, y] + [y, A_C y] = -\|D^*y\|^2 \tag{3.28}$$

for every y in $D(A_C)$, where $D^* = B^*Q^{-1}$.

Therefore $T(t)$ is strongly stable and similar to a contraction $C(t)$ whose generator satisfies (3.28). Moreover,

Corollary 3.3.1 *Let $T(t)$ be a C_0 semigroup with generator A in H. Suppose that*

$$[Ax, x] + [x, Ax] = -\|B^*x\|^2 \quad , x \in D(A),$$

for some linear bounded operator B. Then $T(t)$ is strongly stable if and only if there exists $\alpha > 0$ such that for every x in H

$$\alpha\|x\|^2 \leq \int_0^\infty \|B^*T(t)x\|^2 dt < \infty .$$

Proof: One half of the Corollary is already stated in Theorem 3.3.3. To prove the other half, first, observe that A is dissipative, hence $T(t)$ is a contraction. Therefore, since $T(t)$ is strongly stable, we have, for $x \in D(A)$,

$$\|x\|^2 = \int_0^\infty \|B^*T(t)x\|^2 dt$$

which can be extended to all of H since B is bounded, $T(t)$ is a contraction and $D(A)$ is dense. This completes the proof. □

We now consider the other extension of the Lyapunov equation. In this case, the operator P satisfies

$$[PAx, x] + [x, PAx] = -\|x\|_n^2, \quad x \in D(A)$$

where, this time, the new norm is stronger that the original norm, i.e.,

$$k_1\|x\| \leq \|x\|_n \tag{3.29}$$

for some $k_1 > 0$ and for all x. If we also assume that H equipped with the new norm is complete, then there exists a unique self-adjoint operator V such that

$$\|x\|_n^2 = [Vx, x]$$

for all x [30, 31]. Note that V may be unbounded but because of (3.29) it has an (possibly unbounded) inverse.

We also note that Theorem 3.3.1 which gives a necessary and sufficient condition for the existence of a linear bounded solution $P \geq 0$, still holds. We now prove that the existence of the solution is sufficient for strong stability in the case of uniformly bounded semigroups.

Theorem 3.3.4 *Let $T(t)$, $t \geq 0$, be a uniformly bounded semigroup on H with generator A. Suppose there exists a linear bounded nonnegative operator P satisfying*

$$[PAx, x] + [x, PAx] = -\|x\|_n^2, \quad x \in D(A), \tag{3.30}$$

where $\|\cdot\|_n$ is a new norm on H, and

$$k_1\|x\| \leq \|x\|_n$$

for some $k_1 > 0$. Then, $T(t)$ is strongly stable.

Proof: We know from Theorem 3.3.1 that if there exists a solution to (3.30) then

$$\int_0^\infty \|T(t)x\|_n^2 dt < \infty$$

for all x in $D(A)$. Since $k_1\|T(t)x\| \leq \|T(t)x\|_n$, we have

$$k_1^2 \int_0^\infty \|T(t)x\|^2 dt \leq \int_0^\infty \|T(t)x\|_n^2 dt < \infty .$$

Now, if for an x

$$\int_0^\infty \|T(t)x\|^2 dt < \infty$$

then, $T(t)x \to 0$ as $t \to \infty$ [11]. Therefore, $T(t)$ is strongly stable on $D(A)$ and since $D(A)$ is dense and $T(t)$ is uniformly bounded, we conclude that $T(t)$ is strongly stable on H. □

Example 3.3.1 Let $\{\phi_n, n = 1, 2, \ldots\}$ be an orthonormal basis for H and let

$$Ax = \sum_{n=-\infty}^{\infty} (-n^2 - \frac{1}{(n^2+1)^3}) [x, \phi_n]\phi_n \; .$$

Then, since

$$\text{Re. } [Ax, x] = -(n^2 + \frac{1}{(n^2+1)^3}) \leq 0$$

A generates a contraction semigroup $T(t)$, $t \geq 0$.

Let V be defined by

$$Vx = \sum_{n=-\infty}^{\infty} 2(\frac{n^2}{n^2+1} + \frac{1}{(n^2+1)^4}) [x, \phi_n]\phi_n \; .$$

Then

$$[Vx, x] \geq \sum_{n=-\infty}^{\infty} |[x, \phi_n]|^2 = \|x\|^2 \quad \text{, for } x \in H \; .$$

Therefore V defines a new norm $\|\cdot\|_n$ on H:

$$\|x\|^2 \leq \|x\|_n^2 = [Vx, x] \; .$$

The operator

$$Px = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} [x, \phi_n]\phi_n$$

is self-adjoint, nonnegative, and satisfies the Lyapunov equation

$$2 \text{ Re. } [PAx, x] = -\|x\|_n^2 = -[Vx, x] \; . \tag{3.31}$$

Therefore, from Theorem 3.3.4, we conclude that A generates a strongly stable contraction semigroup $T(t)$. We find

$$T(t)x = \sum_{n=-\infty}^{\infty} e^{-(n^2+\frac{1}{(n^2+1)^3})t} [x, \phi_n]\phi_n \; .$$

Moreover, it is actually exponentially stable since $\|T(t)x\| \leq e^{-t}\|x\|$.

We could have reached the same conclusion directly from the Lyapunov equation (3.31). Since V is a bounded operator, $\|\cdot\|_n^2 = [V\cdot,\cdot]$ is actually an equivalent norm on H and in this case Datko showed that (3.31) implies exponential stability [12].

$\triangle$

Example 3.3.2 Let $H = \overline{\text{span}\{\phi_n\}_{n=1}^{\infty}}$ where $\{\phi_n\}$ are orthonormal. Let

$$Ax = \sum_{n=1}^{\infty} -\frac{1}{n}[x,\phi_n]\phi_n \quad .$$

Then A generates a C_0 semigroup $T(t)$, $t \geq 0$, given by

$$T(t)x = \sum_{n=1}^{\infty} e^{-\frac{t}{n}}[x,\phi_n]\phi_n \quad .$$

We have

$$\begin{aligned}\|T(t)x\|^2 &= \sum_{n=1}^{\infty} e^{-\frac{2t}{n}}|[x,\phi_n]|^2 \\ &\leq \|x\|^2 \quad .\end{aligned}$$

Hence, $T(t)$ is a contraction semigroup. Moreover, for all $x \in H$,

$$\lim_{t\to\infty}\|T(t)x\|^2 = \lim_{t\to\infty}\sum_{n=1}^{\infty} e^{-\frac{2t}{n}}|[x,\phi_n]|^2 = 0$$

or, $T(t)$ is strongly stable.

Note that since $\|T(t)x\| = 1$ (see Example 3.1.1), $T(t)$ is <u>not</u> exponentially stable.

We define the linear bounded operator B by

$$Bx = \sum_{n=1}^{\infty}\sqrt{\frac{2}{n}}[x,\phi_n]\phi_n \quad .$$

Then B is self-adjoint and

$$\begin{aligned}2\,\text{Re.}\,[Ax,x] &= 2\sum_{n=1}^{\infty} -\frac{1}{n}|[x,\phi_n]|^2 \\ &= -\|B^*x\|^2 \quad ,\end{aligned}$$

which shows that A satisfies the condition of Corollary 3.3.1. $\triangle$

Chapter 4

Stabilization

In this chapter, we study feedback stabilizability of the distributed control sytems described by the differential equation

$$\dot{x}(t) = Ax(t) + Bu(t) \ . \tag{4.1}$$

Such a system is denoted by the pair (A, B). Unless otherwise stated, B is a linear bounded operator from the control space U to the state space H.

Our problem is to find controls of the form $u = Fx$, where F is a linear bounded feedback operator, such that the "closed loop" system

$$\dot{x}(t) = (A + BF)x(t)$$

is strongly stable. In this sense, stabilizability of (A, B) is equivalent to the stability of the closed loop system. Results obtained in Chapter 3 will play a key role in what follows.

Here, we again focus on contractive systems where A generates a contraction semigroup. We apply the feedback control $u = -B^*x$ and investigate the stability

of the semigroup generated by $A - BB^*$. The reason for this choice of feedback is twofold. First, the closed loop system $\dot{x}(t) = (A - BB^*)x(t)$ is also contractive. Hence our earlier results on contraction semigroups apply. But, any F such that Re.$[BFx, x] \leq 0$ will preserve contractivity. However, from practical point of view, $-B^*$ does not require any computations hence it is robust, and this is the second reason for its choice.

This chapter is organized in three sections. First, we apply the state space decomposition developed in Section 3.2 to obtain conditions for strong stability of the semigroup generated by $A - BB^*$. Limitations of the feedback $-B^*$ for strong stabilization are also discussed. Then, we apply the extensions of Lyapunov equation developed in Section 3.3. Finally, we show that this approach leads to interesting results concerning a Riccati equation for contractive systems.

4.1 Stabilization via Decomposition of the State Space

Let $T(t)$ be the contraction semigroup associated with the contractive system (A, B). Let $u = -B^*x$ be the feedback control. Then, the closed loop system

$$\dot{x}(t) = (A - BB^*)x(t)$$

is also contractive by the fact that $-BB^*$ is bounded dissipative and A is maximal dissipative.

We know that the feedback $-B^*$ will weakly stabilize the contractive system (A, B) if its weakly unstable states are controllable [5] (see also Chapter 2, The-

orem 2.2.8). Here, we will explore strong stabilization with the same feedback. Specifically, we will apply the state space decomposition developed in Chapter 3 to the closed loop system.

We note that in general, application of the theory developed in Chapter 3 for stabilization, particularly the state space decomposition, is not only limited to feedbacks of the form $-B^*$. In fact, the feedback operator F need not be dissipative. As long as the closed loop operator $A+BF$ generates a contraction semigroup, the state space can be decomposed with respect to this contraction semigroup. Conditions for strong stability can then be found.

Let $Z(t)$, $t \geq 0$, be the contractive semigroup generated by $\mathcal{A} = A - BB^*$. Let $H_i(Z)$ be the isometric subspace of $Z(t)$:

$$H_i(Z) = \{x \in H : \|Z(t)x\| = \|x\| \ \ , t \geq 0\}$$

and let

$$M_s(Z) = \{x \in H : Z(t)x \to 0 \ \text{ as } \ t \to \infty\}$$

be its strongly stable subspace. Also, let C_z be defined by

$$C_z^2 = \lim_{t\to\infty} Z(t)^* Z(t) \ \ ,$$

then it is a nonnegative contraction. Therefore, applying Theorem 3.2.2 of Chapter 3, we have the following decomposition of the state space H with respect to the semigroup $Z(t)$

$$H = H_i(Z) \oplus M_z \oplus M_s(Z) \ \ . \tag{4.2}$$

The feedback system $(A - BB^*, B)$ or, equivalently, the semigroup $Z(t)$, $t \geq 0$, is strongly stable if and only if $H = M_s(Z)$. In other words, the contractive system

(A, B) is strongly stabilizable by the feedback $-B^*$ if and only if $H_i(Z) \oplus M_z = \{0\}$. This means that we must have $H_i(Z) = \{0\}$ and $M_z = \{0\}$.

It is evident that stabilization of $T(t)$ via the decomposition (4.2) is a two step procedure. First, we require the subspace M_z to be trivial, for which it is necessary and sufficient that the operator C_z be a projection (see Chapter 3, Lemma 3.2.2).

We have seen in Chapter 3 several conditions for C_z to be a projection. If $Z(t)$ is self-adjoint or if the generator $\mathcal{A}$ has compact resolvent then C_z is a projection. These conditions are satisfied when the original semigroup $T(t)$ with generator A has the same properties. Because, if $T(t)$ is self-adjoint then so is its generator A. Hence $\mathcal{A} = A - BB^*$ is also self-adjoint and generates a self-adjoint semigroup $Z(t)$. In the same way, if A has compact resolvent then so does $A - BB^*$ since B is linear bounded. Also, C_z is a projection when $Z(t)$ is normal.

It is interesting to note that a fourth condition which will ensure C_z to be a projection, namely that $Z(t)$ be a co-isometry, cannot be satisfied. Because, if $Z(t)$ is a co-isometry, i.e., $Z(t)^*$ is an isometry, then

$$\text{Re. } [(A - BB^*)^* x, x] = 0 \quad , x \in D(A) \, .$$

Or, Re. $[A^* x, x] = \|B^* x\|^2$. Since Re. $[A^* x, x] \leq 0$ and $\|B^* x\|^2 \geq 0$, we must have $B^* x = 0$ for x in $D(A)$. Hence, $D(A)$ being dense in H, this implies $B^* = 0$, in which case there is no feedback.

Next, we investigate conditions for $Z(t)$ to be completely non-isometric, or $H_i(Z) = \{0\}$.

Proposition 4.1.1 *Let $T(t)$, $t \geq 0$, be a contraction semigroup with generator A on H and $B : U \to H$ be a linear bounded operator. Then, $A - BB^*$ generates a*

completely non-isometric contraction semigroup $Z(t)$ (say) if and only if $H_i(T)$ is (A^*, B) controllable, i.e., $H_i(T) \subseteq M_c(A^*, B)$.

Proof: Suppose $x \in H_i(Z)$. Then, $\|Z(t)x\| = \|x\|$ for $t \geq 0$. Differentiating with respect to t, we have for $x \in H_i(Z) \bigcap D(A)$

$$\text{Re. } [(A - BB^*)Z(t)x, Z(t)x] = 0 \ .$$

Or,

$$\text{Re. } [AZ(t)x, Z(t)x] = \|B^* Z(t)x\|^2 \ .$$

Therefore, since A is dissipative, $\|B^* Z(t)x\| = 0$ for $t \geq 0$, which, since $H_i(Z) \bigcap D(A)$ is dense in $H_i(Z)$, shows that $x \in M_{uc}((A - BB^*)^*, B) = M_{uc}(A^*, B)$. Hence

$$H_i(Z) \subseteq M_{uc}(A^*, B) \ .$$

The identity [23]

$$Z(t)x = T(t)x - \int_0^t T(t-\sigma)BB^* Z(\sigma)x d\sigma \tag{4.3}$$

shows that for $x \in M_{uc}(A^*, B)$, $Z(t)x = T(t)x$. Hence, if x is in $H_i(Z)$, then $\|Z(t)x\| = \|T(t)x\| = \|x\|$, which shows that $H_i(Z) \subseteq H_i(T)$. Therefore,

$$H_i(Z) \subseteq H_i(T) \bigcap M_{uc}(A^*, B) \ .$$

Conversely, if $x \in H_i(T) \bigcap M_{uc}(A^*, B)$, then $Z(t)x = T(t)x$ by (4.3) and $\|T(t)x\| = \|x\| = \|Z(t)x\|$. Consequently,

$$H_i(Z) = H_i(T) \bigcap M_{uc}(A^*, B) \ ,$$

and $Z(t)$ is completely non-isometric if and only if $H_i(T) \bigcap M_{uc}(A^*, B) = \{0\}$. $\square$

We summarize the result on strong stabilization with the feedback $-B^*$ below:

Theorem 4.1.1 *Suppose for a contractive system (A, B) either A generates a self-adjoint semigroup $T(t)$ or A has compact resolvent. Then, the system $\dot{x} = Ax + Bu$ is strongly stabilizable by the feedback $u = -B^*x$ if and only if the isometric subspace $H_i(T)$ of $T(t)$ is (A^*, B) controllable.*

The above Theorem is analogous to Benchimol's result on weak stabilization via the same feedback $-B^*$. In our case, the isometric subspace $H_i(T)$ may be regarded as the (strongly) unstable subspace. Although this is not true for general contraction semigroups, we have shown that for those with compact resolvent or for self-adjoint semigroups, it is indeed the case. Hence, controllability of the unstable subspace plays an important role in strong stabilizability of contractive systems as it did in weak stabilizability. Another interesting point to note is that in case of strong stabilization, controllability of (A^*, B) comes into play rather than that of (A, B). This also happens in weak stabilization if one replaces weak stability of $T(t)$ by that of $T(t)^*$, which are equivalent.

State space decomposition approach also points out the limitations of using the feedback $-B^*$ for strong stabilization of contractive systems. It is always restricted to the class of (feedback) semigroups for which the operator C_z is a projection. The problem to be resolved then, is to characterize such semigroups, which was the case in the study of strong stability of contraction semigroups as well.

Example 4.1.1 To demonstrate stabilization via the state space decomposition, we take the semigroup of Example 3.2.1 of Chapter 3 where

$$H = \overline{\text{span}\{\phi_n\}_{n=1}^{\infty}} \oplus \overline{\text{span}\{\psi_m\}_{m=1}^{\infty}}$$

and

$$T(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n+i\beta_n)t}[x,\phi_n]\phi_n + \sum_{m=1}^{\infty} e^{i\gamma_m t}[x,\psi_m]\psi_m \quad .$$

$\{\phi_n\}$ and $\{\psi_m\}$ jointly form an orthonormal basis for H, and $\alpha_n, \beta_n, \gamma_m$ are real and $\alpha_n > 0$.

We have seen that $T(t)$, $t \geq 0$, is a normal contractive semigroup with the isometric subspace

$$H_i(T) = \overline{\text{span}\{\psi_m\}_{m=1}^{\infty}} \quad .$$

The generator A of $T(t)$ is given by

$$Ax = \sum_{n=1}^{\infty}(-\alpha_n + i\beta_n)[x,\phi_n]\phi_n + \sum_{m=1}^{\infty} i\gamma_m [x,\psi_m]\psi_m \quad .$$

Let

$$Bx = \sum_{m=1}^{\infty} b_m [x,\psi_m]\psi_m \tag{4.4}$$

where b_m are real. Then B is self-adjoint. With the feedback $-B^*$ we have

$$(A - BB^*)x = \sum_{n=1}^{\infty}(-\alpha_n + i\beta_n)[x,\phi_n]\phi_n + \sum_{m=1}^{\infty}(-b_m^2 + i\gamma_m)[x,\psi_m]\psi_m$$

which generates a contraction semigroup $Z(t)$ given by

$$Z(t)x = \sum_{n=1}^{\infty} e^{(-\alpha_n+i\beta_n)t}[x,\phi_n]\phi_n + \sum_{m=1}^{\infty} e^{(-b_m^2+i\gamma_m)t}[x,\psi_m]\psi_m \quad .$$

Since

$$\|Z(t)x\|^2 = \|Z(t)^*x\|^2 = \sum_{n=1}^{\infty} e^{-2\alpha_n t}\,|[x,\phi_n]|^2 + \sum_{m=1}^{\infty} e^{-2b_m^2 t}\,|[x,\psi_m]|^2 \, ,$$

$Z(t)$ is also normal. Hence the operator C_z is a projection.

Therefore, according to Theorem 4.1.1, $Z(t)$ is strongly stable if and only if $H_i(T) \bigcap M_{uc}(A^*, B) = \{0\}$.

We have

$$\begin{aligned} \|B^*T(t)x\|^2 &= \|\sum_{m=1}^{\infty} b_m\, e^{i\gamma_m t}[x,\psi_m]\psi_m\|^2 \\ &= \sum_{m=1}^{\infty} b_m^2\, \|[x,\psi_m]\|^2 \ . \end{aligned}$$

Hence,

$$\begin{aligned} M_{uc}(A^*,B) &= \bigcap_{t\geq 0} \mathbf{N}(B^*T(t)) \\ &= \overline{\text{span}\{\phi_n\}_{n=1}^{\infty}} \oplus \overline{\text{span}\{\psi_m,\ \text{for } m \text{ s.t. } b_m=0\}} \ . \end{aligned}$$

Then,

$$H_i(T)\bigcap M_{uc}(A^*,B) = \overline{\text{span}\{\psi_m,\ \text{for } m \text{ s.t. } b_m=0\}} \ .$$

We then conclude that the feedback $-B^*$ strongly stabilizes the system if and only if $b_m \neq 0$ for any m in (4.4). This also shows that a finite dimensional feedback cannot stabilize the system. $\triangle$

Example 4.1.2 Consider the distributed system

$$\begin{aligned} \frac{\partial x(t,\xi)}{\partial t} &= -\frac{\partial x(t,\xi)}{\partial \xi} + Bu \quad , \xi \in [0,2\pi]\ , t>0 \\ x(t,0) &= x(t,2\pi) \ . \end{aligned}$$

If we take $H = L^2[0,2\pi]$ and $A = -\frac{\partial}{\partial\xi}$ with

$$D(A) = \{x \in H : x \text{ absolutely continuous, } x' \in H, x(0) = x(2\pi)\} \ ,$$

then we have the abstract equation

$$\dot{x}(t) = Ax(t) + Bu \ .$$

We note that A has compact resolvent, $A = -A^*$, and it generates a unitary semigroup $T(t)$

$$T(t)x = \sum_{n=-\infty}^{\infty} e^{int}[x,\phi_n]\phi_n$$

where $\phi_n(\xi) = e^{in\xi}/\sqrt{2\pi}$ form an orthonormal basis for the space H.

Since $T(t)$ is unitary it is unstable and $H = H_u(T)$. Let

$$Bx = \sum_{n=-\infty}^{\infty} b_n [x, \phi_n]\phi_n$$

where b_n are real ($B = B^*$) and $b_n \neq 0$ for any n. Also suppose

$$\lim_{n\to\infty} |b_n| = 0 \ ,$$

then B is compact.

Since

$$\begin{aligned} \|B^*T(t)x\|^2 &= \| \sum_{n=-\infty}^{\infty} b_n e^{int} [x, \phi_n]\phi_n \|^2 \\ &= \sum_{n=-\infty}^{\infty} b_n^2 |[x, \phi_n]|^2 \ , \end{aligned}$$

we have $B^*T(t)x = 0$ for $t \geq 0 \Leftrightarrow x = 0$, hence (A^*, B) is controllable. Therefore, according to Theorem 4.1.1, since A has compact resolvent $A - BB^*$ generates a strongly stable semigroup $Z(t)$.

Note that $Z(t)$ has the form

$$Z(t)x = \sum_{n=-\infty}^{\infty} e^{(-b_n^2+in)t} [x, \phi_n]\phi_n$$

and since $\lim_{n\to\infty} |b_n| = 0$, $Z(t)$ is <u>not</u> exponentially stable. $\triangle$

Example 4.1.3 We have shown in Example 3.2.2 that the heat equation on a bounded domain is unstable. Here, we will stabilize the controlled system

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{\partial^2 x}{\partial \xi^2} + Bu \\ &= Ax + Bu \ . \end{aligned}$$

Let $B = I$ and $u = -B^*x$. Then, $A - BB^* = A - I$. Since

$$B^*T(t)x = T(t)x = 0 \ \text{ for } t \geq 0 \Longleftrightarrow x = 0 \ ,$$

(A^*, B) is controllable. Also, A is self-adjoint. Therefore, by Theorem 4.1.1, the semigroup $Z(t)$ generated by $A - I$ is strongly stable.

We note that since A has the expansion

$$Ax = \sum_{n=-\infty}^{\infty} -n^2 [x, \phi_n] \phi_n \quad ,$$

the feedback semigroup is given by

$$Z(t)x = \sum_{n=-\infty}^{\infty} e^{-(n^2+1)t} [x, \phi_n] \phi_n \quad .$$

$Z(t)$ is such that

$$\|Z(t)x\| \leq e^{-t} \|x\| \quad ,$$

hence it is exponentially stable. This is due to the fact that $Z(t)$ is a compact semigroup and for compact semigroups weak, strong and exponential stabilities are all equivalent. $\triangle$

4.2 Application of Lyapunov Equation

In Section 3.3 of Chapter 3 we gave sufficient conditions for a uniformly bounded C_0 semigroup to be weakly and strongly stable by extending the well known Lyapunov equation. Clearly, the results hold for contraction semigroups, which are special cases of uniformly bounded semigroups. In this section we focus on a contractive system (A, B) and apply the abovementioned results to investigate conditions for stabilizability via the feedback $-B^*$.

Since A is dissipative and B is linear bounded, the semigroup $Z(t)$ generated by $\mathcal{A} = A - BB^*$ is a contraction. We have, for $x \in D(A)$,

$$\frac{d}{dt} \|Z(t)x\|^2 = 2 \text{ Re. } [\mathcal{A}Z(t)x, Z(t)x] \, .$$

Integrating from 0 to t and letting $t \to \infty$, we see that $Z(t)$ satisfies the following equation

$$\begin{aligned} \|x\|^2 &= \lim_{t\to\infty} \|Z(t)x\|^2 - \int_0^\infty 2\ \mathrm{Re.}\ [\mathcal{A}Z(t)x, Z(t)x]dt \qquad (4.5) \\ &= \lim_{t\to\infty} \|Z(t)x\|^2 - \int_0^\infty 2\ \mathrm{Re.}\ [AZ(t)x, Z(t)x]dt + 2\int_0^\infty \|B^* Z(t)x\|^2 dt \end{aligned}$$

for x in $D(\mathcal{A}) = D(A)$.

Define the linear self-adjoint operator P by

$$[Px, x] = \int_0^\infty \|B^* Z(t)x\|^2 dt \ \ , x \in H \ . \qquad (4.6)$$

Then, P is nonnegative and

$$[Px, x] \leq \frac{1}{2}\|x\|^2 \qquad (4.7)$$

from (4.5). Replacing x by $Z(t)x$ in (4.6) and differentiating with respect to t yield

$$2\ \mathrm{Re.}\ [P\mathcal{A}Z(t)x, Z(t)x] = -\|B^* Z(t)x\|^2 \ \ x \in D(\mathcal{A}) \ .$$

Seting $t = 0$, we see that P satisfies the Lyapunov equation

$$2\ \mathrm{Re.}\ [P(A - BB^*)x, x] = -\|B^* x\|^2 \ \ , x \in D(A) \ , \qquad (4.8)$$

which was investigated in Section 3.3. Therefore, we have the following result on the existence of the solution P of equation (4.8):

Lemma 4.2.1 *Let A be the generator of a contractive semigroup $T(t)$, $t \geq 0$, and B a linear bounded operator. Then, there always exists a nonnegative operator P satisfying the Lyapunov equation*

$$2\ \mathrm{Re.}\ [P(A - BB^*)x, x] = -\|B^* x\|^2 \ \ , x \in D(A) \ .$$

Note that the Lemma holds for any linear bounded B, including compact operators. We can now apply Theorem 3.3.2 to obtain conditions for the semigroup $Z(t)$ generated by $A - BB^*$ to be weakly stable. According to the above Lemma, the first condition of the Theorem (existence of P) is always satisfied. If, in addition, $\mathbf{N}(B^*) = \{0\}$ then $\|B^*x\|(= \|x\|_n)$ defines a new norm with the property

$$\|x\|_n \leq \|B^*\| \ \|x\| \ ,$$

in which case $Z(t)$ is weakly stable. But, for weak stability, we can prove a weaker condition.

Theorem 4.2.1 *Let (A, B) be a contractive system. If the pair (A^*, B) is controllable, then $A - BB^*$ generates a weakly stable contraction semigroup $Z(t)$.*

Proof. As before, define $P \geq 0$ by (4.6) and let $x \in \mathbf{N}(P)$. Then

$$[Px, x] = \int_0^\infty \|B^* Z(t)x\|^2 dt = 0 \ ,$$

which implies $B^*Z(t)x = 0$, $t \geq 0$, or $x \in M_{uc}((A - BB^*)^*, B)$. Since $M_{uc}((A - BB^*)^*, B) = M_{uc}(A^*, B)$, we have $\mathbf{N}(P) \subseteq M_{uc}(A^*, B)$. Conversely, let $x \in M_{uc}(A^*, B)$. Then $B^*Z(t)x = 0$, $t \geq 0$, or $x \in \mathbf{N}(P)$. Hence,

$$\mathbf{N}(P) = M_{uc}(A^*, B) \ .$$

Consider now, for x in H,

$$[PZ(t)x, Z(t)x] = \int_t^\infty \|B^* Z(\sigma)x\|^2 d\sigma \ .$$

Then,

$$\lim_{t \to \infty} [PZ(t)x, Z(t)x] = 0 \ . \tag{4.9}$$

Therefore, if $M_{uc}(A^*, B) = \{0\}$ then $P > 0$, and since $Z(t)$ is a contraction, (4.9) implies that $Z(t)$ is weakly stable. □

For strong stability, we have the following:

Theorem 4.2.2 *If there exists $\alpha > 0$ such that for every x in H*

$$\alpha\|x\|^2 \leq \int_0^\infty \|B^*Z(t)x\|^2 dt$$

then $Z(t)$ is strongly stable.

Proof: The proof is all but trivial. The condition in the Theorem is equivalent to $[Px, x] \geq \alpha\|x\|^2$, in which case P defines an equivalent norm, and (4.9) implies strong stability. □

Recall that a system (A, B) is conservative if Re. $[Ax, x] = 0$ for all x in $D(A)$. We now prove a generalization of Corollary 3.3.1 of Chapter 3 which also applies to strong stabilization of conservative systems.

Theorem 4.2.3 *Suppose A is such that*

$$-\delta\|B^*x\|^2 \leq \text{Re. } [Ax, x] \leq 0 \quad , x \in D(A) \tag{4.10}$$

for some $\delta \geq 0$. Then $A - BB^*$ generates a strongly stable contractive semigroup $Z(t)$ if and only if there exists $\alpha > 0$ such that

$$\alpha\|x\|^2 \leq \int_0^\infty \|B^*Z(t)x\|^2 dt \quad , x \in H \ .$$

Proof: One half of the theorem is already given in Theorem 4.2.2. To prove the other half, we rewrite (4.10) as

$$-\gamma\|B^*x\|^2 \leq \text{Re. } [(A - BB^*)x, x] \tag{4.11}$$

where $\gamma = \delta + 1$.

Since $A - BB^*$ is dissipative, the integral

$$\int_0^\infty \|B^* Z(t)x\|^2 dt$$

exists for $x \in D(A)$ and, by extension, for all x. Therefore, replacing x by $Z(t)x$ and integrating both sides of (4.11), we have

$$-\gamma \int_0^\infty \|B^* Z(t)x\|^2 dt \leq \lim_{t \to \infty} \|Z(t)x\|^2 - \|x\|^2 .$$

If $Z(t)$ is strongly stable,

$$\frac{1}{\gamma}\|x\|^2 \leq \int_0^\infty \|B^* Z(t)x\|^2 dt$$

which proves the theorem. □

Finally, we have the following result for exponential stabilization.

Theorem 4.2.4 *If there exists $\alpha > 0$ such that*

$$\alpha\|x\| \leq \|B^* x\| \tag{4.12}$$

for all $x \in H$, then $A - BB^*$ generates an exponentially stable contractive semigroup.

Proof. Since B^* is already bounded, (4.12) implies that

$$\|x\|_n^2 = [BB^* x, x]$$

defines an equivalent norm. In this case, the Lyapunov equation (4.8) satisfies Datko's theorem on exponential stability (Chapter 2, Theorem 2.2.6), and $Z(t)$ is exponentially stable. □

Example 4.2.1 In Example 4.1.3, we have shown that the heat equation is exponentially stabilized by the feedback $-I$ (identity). According to Theorem 4.2.4 with $B = I$ this result is as expected.

Now, let

$$Bx = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}[x,\phi_n]\phi_n \quad .$$

Then

$$Px = \sum_{n=-\infty}^{\infty} \frac{1}{2(1+n^2(n^2+1)^2)}[x,\phi_n]\phi_n$$

satisifies the Lyapunov equation (4.8).

Note that B is self-adjoint but it is not bounded from below. Therefore Theorem 4.2.4 does not apply. However, since

$$B^*T(t)x = \sum_{n=-\infty}^{\infty} \frac{e^{-n^2 t}}{n^2+1}[x,\phi_n]\phi_n \quad ,$$

we have

$$B^*T(t)x = 0 \;\text{ for }\; t \geq 0 \Longleftrightarrow x = 0 \quad ,$$

which shows that the pair (A^*, B) is controllable. Hence, from Theorem 4.2.1, $A - BB^*$ generates a weakly stable contractive semigroup $Z(t)$. It is given by

$$Z(t)x = \sum_{n=-\infty}^{\infty} e^{-(n^2+\frac{1}{(n^2+1)^2})t}[x,\phi_n]\phi_n \quad ,$$

and is compact. Therefore weak stability implies exponential stability and the system is again exponentially stabilized by the feedback $-B^*$. △

Example 4.2.2 Suppose $H = \overline{\text{span}\{\phi_n\}_{n=1}^{\infty}}$ where $\{\phi_n\}$ are orthonormal. As in Example 3.3.2, let

$$\begin{aligned} Ax &= \sum_{n=1}^{\infty} -\frac{1}{n}[x,\phi_n]\phi_n \\ Bx &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}[x,\phi_n]\phi_n \quad . \end{aligned}$$

Then

$$(A - BB^*)x = \sum_{n=1}^{\infty} -\frac{3}{n}[x, \phi_n]\phi_n \quad ,$$

and $P = \frac{1}{3}I$ satisfies the Lyapunov equation (4.8). $A - BB^*$ generates the semigroup

$$Z(t)x = \sum_{n=1}^{\infty} e^{-\frac{3}{n}t}[x, \phi_n]\phi_n$$

which is clearly strongly stable. In order to demonstrate Theorem 4.2.2, we observe that

$$\begin{aligned}\int_0^\infty \|B^* Z(t)x\|^2 dt &= \int_0^\infty \sum_{n=1}^{\infty} \frac{2}{n} e^{-\frac{6}{n}t} \|[x, \phi_n]\|^2 \; dt \\ &= \frac{1}{3}\|x\|^2 \quad .\end{aligned}$$

Note that $Z(t)$ is not exponentially stable. △

Example 4.2.3 Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal basis for H. Let the self-adjoint operators A and B be defined by

$$\begin{aligned}Ax &= \sum_{n=-\infty}^{\infty} -n^2 [x, \phi_n]\phi_n \\ Bx &= \sum_{n=-\infty}^{\infty} \sqrt{2 - \frac{2}{(n^2+1)} + \frac{2}{(n^2+1)^4}}\,[x, \phi_n]\phi_n \quad .\end{aligned}$$

Then, since A is dissipative and

$$\|B^* x\| \geq \|x\|$$

according to Theorem 4.2.4, $A - BB^*$ generates an exponentially stable contraction semigroup. △

4.3 Steady State Riccati Equation for Contractive Systems

We have shown in the previous section that the Lyapunov equation (4.8) associated with the closed loop system $(A - BB^*, B)$ always has a nonnegative solution. This result has further implications concerning a related linear quadratic regulator (LQR) problem. The LQR problem can be stated as follows.

Consider the distributed system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ x(0) &= x_0 \in H \end{aligned} \tag{4.13}$$

where A generates a contraction semigroup $T(t)$, B is a linear bounded operator and $u(\cdot) \in L^2([0,\infty);U)$. The LQR problem is to find $u(\cdot)$ in $L^2([0,\infty);U)$ which will minimize the cost functional

$$J = \int_0^\infty [BB^*x(t), x(t)]dt + \int_0^\infty \|u(t)\|^2 dt \,. \tag{4.14}$$

It can be shown that if for each initial state x_0 there exists a control $u(\cdot)$ for which J is finite, then the following steady state Riccati equation (SSRE)

$$[QAx, x] + [x, QAx] - \|B^*Qx\|^2 = -[BB^*x, x] \quad , x \in D(A)$$

has a bounded nonnegative solution Q [3, 71]. The feedback control $u(t) = -B^*Qx(t)$ is the optimal control which minimizes J. If, in addition, BB^* is a strictly positive operator then the control exponentially stabilizes the system (4.13). In the following discussion, unless otherwise stated, we only assume B to be a linear bounded operator.

Now, let $u(t) = -B^*x(t)$ in (4.13). Then the closed loop system $\dot{x}(t) = (A - BB^*)x(t)$, $x(0) = x_0$, has the solution

$$x(t) = Z(t)x_0$$

where, as before, $Z(t)$ is the contraction semigroup generated by $A - BB^*$. For this choice of control we find

$$\begin{aligned} J &= \int_0^\infty [BB^*x(t), x(t)]dt + \int_0^\infty \|u(t)\|^2 dt \\ &= \int_0^\infty [BB^*Z(t)x_0, Z(t)x_0]dt + \int_0^\infty \|B^*Z(t)x_0\|^2 dt \\ &= 2\int_0^\infty \|B^*Z(t)x_0\|^2 dt\,. \end{aligned} \tag{4.15}$$

Therefore

$$J \leq \|x_0\|^2 \quad \text{for all } x_0 \in H, \tag{4.16}$$

by (4.6) and (4.7). We conclude that for the contractive system (4.13), there always exists a control $u(\cdot)$ which will make the cost functional finite for every state $x_0 \in H$. We then have:

Theorem 4.3.1 *Let A be the generator of a contraction semigroup and B a linear bounded operator. Then the SSRE*

$$[QAx, x] + [x, QAx] - \|B^*Qx\|^2 + [BB^*x, x] = 0 \quad , x \in D(A) \tag{4.17}$$

always has a bounded nonnegative solution Q.

This Theorem shows that the SSRE (4.17) in which A is dissipative and B is any linear bounded operator always has a solution. The question of existence of such a nonnegative solution Q for an SSRE in which B is compact was raised by

Gibson in [22]. Here, we are able to show that for a contractive system (A, B) even when B is compact (4.17) does admit a nonnegative solution.

The question raised by Gibson in [22] was in connection with the problem of stabilization via compact feedback. He showed that a strongly stable contractive system cannot be exponentially stabilized by compact feedback. We must note that although (4.17) admits a nonnegative solution Q, the control $u(t) = -B^*Qx(t)$ need not be a stabilizing control.

Since the control $u(t) = -B^*Qx(t)$ is optimal, the cost functional (4.14) attains its minimum value with this control. Let J_{min} denote the corresponding minimum cost. Then we know from LQR theory that

$$J_{min} = [Qx_0, x_0] \geq 0$$

where Q is the nonnegative solution of the SSRE (4.17) and x_0 is the initial state [3].

The feedback control $u(t) = -B^*x(t)$ was investigated in the previous section. The cost functional for this control is given by (4.15). It can be rewritten as

$$J = 2\,[Px_0, x_0]$$

where P satisfies the Lyapunov equation

$$2 \text{ Re. } [P(A - BB^*)x, x] = -\|B^*x\|^2 \quad , x \in D(A) \ .$$

Therefore, from (4.16),

$$0 \leq [Qx_0, x_0] \leq 2\,[Px_0, x_0] \leq \|x_0\|^2 \ .$$

We summarize the result below.

Theorem 4.3.2 *Let A be the generator of a contraction semigroup and B a linear bounded operator. Then the solution Q of the SSRE (4.17) is such that*

$$0 \le [Qx, x] \le \|x\|^2 \quad , x \in H \ .$$

In general, if Q is the nonnegative solution of the SSRE, then $A - BB^*Q$ genarates a C_0 semigroup which is quasi affine transform of a contraction semigroup [43]. When Q is strictly positive, the semigroup generated by $A - BB^*Q$ is similar to a contraction. In order for $A - BB^*Q$ to generate a contraction semigroup it is necessary and sufficient that

$$\text{Re.}\,[(A - BB^*Q)x, x] \le 0 \quad , x \in D(A) \ . \tag{4.18}$$

Since A is already dissipative, this holds if the operator $(BB^*Q + QBB^*)$ is nonnegative.

In particular, if the nonnegative operators BB^* and Q commute then the product $QBB^*(= BB^*Q)$ is also nonnegative. In this case $A - BB^*Q$ is dissipative and generates a contraction semigroup.

Now, if for all $x \in H$

$$\alpha\|x\| \le \|B^*x\|$$

or if BB^* is strictly positive (invertible) then, as we have seen in Chapter 2, Theorem 2.2.11, the optimal control $u(t) = -B^*Qx(t)$ is such that the closed loop system

$$\dot{x}(t) = (A - BB^*Q)x(t)$$

is exponentially stable.

We note that, as expected, B cannot be compact in this case. Because, if B is compact, so is B^* and a compact operator can not be bounded from below.

Therefore, when BB^* is strictly positive we have two different feedback controls for exponential stabilization of the contractive system (A, B). The control $u(t) = -B^*Qx(t)$ is a result of the related LQR problem and it is optimal for the cost (4.14). On the other hand, Theorem 4.2.4 shows that $u(t) = -B^*x(t)$ is another control which will ensure exponential stability.

Although both $A-BB^*$ and $A-BB^*Q$ generate exponentially stable semigroups, the feedback semigroup $Z(t)$ generated by $A - BB^*$ is contractive while it is not always the case for $A - BB^*Q$.

It is evident that, when BB^* is invertible, there are advantages of using the feedback $-B^*$ for exponential stabilization of the contractive system (A, B). It does not require solving the SSRE, which is difficult computationally. Also, the feedback system is contractive and hence all the theory developed for contractive semigroups apply.

For conservative systems, i.e., Re. $[Ax, x] = 0$, the feedback control $u(t) = -B^*x(t)$, where B is linear bounded, turns out to be optimal as well.

If A is a conservative operator, the SSRE

$$[QAx, x] + [x, QAx] - \|B^*Qx\|^2 = -[BB^*x, x] \quad , x \in D(A)$$

always admits $Q = I$ (identity) as a (strictly) positive solution, since

$$[Ax, x] + [x, Ax] - \|B^*x\|^2 = -\|B^*x\|^2 \quad , x \in D(A).$$

Note that this solution is valid even for compact B. Therefore, the optimal control for the conservative system is $u(t) = -B^*x(t)$. With this choice of control the system has the solution $x(t) = Z(t)x_0$, $t \geq 0$, where $Z(t)$ is generated by $A - BB^*$

and x_0 is the initial state. The minimum cost is

$$J_{min} = 2 \int_0^\infty \|B^* Z(t) x_0\|^2 dt \ .$$

We also note that in general $Q = I$ may not be the only solution of SSRE (when BB^* is strictly positive it <u>is</u> the unique solution). Also, the degree of stabilization (weak, strong or exponential) by the optimal control $-B^* x(t)$ depends on other conditions as indicated in Theorems 4.2.1, 4.2.2 and 4.2.3.

Example 4.3.1 We illustrate the results of this section by solving the SSRE for the heat equation where

$$Ax = \sum_{n=-\infty}^{\infty} -n^2 [x, \phi_n] \phi_n \ .$$

Let

$$Bx = \sum_{n=-\infty}^{\infty} b_n [x, \phi_n] \phi_n$$

where b_n are real numbers. The SSRE (4.17) can be decomposed as follows.

Let Q be defined by

$$Qx = \sum_{n=-\infty}^{\infty} q_n [x, \phi_n] \phi_n$$

where, since Q is nonnegative, $q_n \geq 0$ are real numbers. Then SSRE becomes

$$\sum_{n=-\infty}^{\infty} (-2n^2 q_n - |b_n|^2 q_n^2 + |b_n|^2) |[x, \phi_n]|^2 = 0 \ ,$$

which, for each n, is a quadratic equation in q_n. Hence, since $q_n \geq 0$,

$$q_n = \frac{n^2 - \sqrt{n^4 + |b_n|^2}}{-|b_n|^2}$$

for n such that $b_n \neq 0$, and $q_n = 0$ otherwise.

For $B = I$, The solution is

$$Qx = \sum_{n=-\infty}^{\infty} (-n^2 + \sqrt{n^4 + 1}) [x, \phi_n] \phi_n \ ,$$

and $[Qx, x] \leq \|x\|^2$ as expected.

Since $BB^*(= I)$ is strictly positive, the optimal control $u(t) = -B^*Qx(t)$ exponentially stabilizes the heat equation. With the optimal control, the closed loop operator $A - BB^*Q$ generates the semigroup $S(t)$:

$$S(t)x = \sum_{n=-\infty}^{\infty} e^{-\sqrt{n^4+1}\ t}[x, \phi_n]\phi_n$$

which is exponentially stable. Moreover, since the operators BB^* and Q commute, $S(t)$, $t \geq 0$, is a contractive semigroup. If x is the initial state the minimum value of the cost associated with the SSRE in question is

$$\begin{aligned} J_{min} &= [Qx, x] \\ &= \sum_{n=-\infty}^{\infty} (-n^2 + \sqrt{n^4+1})\,|[x, \phi_n]|^2 \ . \end{aligned}$$

In Example 4.1.3, we showed that for $B = I$, $A - BB^*$ also generates an exponentially stable contractive semigroup $Z(t)$:

$$Z(t)x = \sum_{n=-\infty}^{\infty} e^{-(n^2+1)t}[x, \phi_n]\phi_n \ .$$

We observe that although for any initial state $x \in H$,

$$\|Z(t)x\| \leq \|S(t)x\| \quad , t \geq 0 \ ,$$

$S(t)x$ is the optimal trajectory for the cost functional (4.14).

Now, as in Example 4.2.1, if we take

$$Bx = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}[x, \phi_n]\phi_n$$

then B is self-adjoint, nonnegative, and since $\lim_{n\to\infty} |1/(n^2+1)| = 0$, it is compact.

SSRE still has a nonnegative solution Q given by

$$Qx = \sum_{n=-\infty}^{\infty} \left(-n^2(n^2+1)^2 + \sqrt{n^4(n^2+1)^4+1}\right)[x,\phi_n]\phi_n \ .$$

$A - BB^*Q$ generates the semigroup

$$S(t)x = \sum_{n=-\infty}^{\infty} e^{-\frac{\sqrt{n^4(n^2+1)^4+1}}{(n^2+1)^2}t}[x,\phi_n]$$

which is again exponentially stable. In other words, the compact feedback $-B^*Q$ exponentially stabilizes the contractive system (A, B). This may seem in violation of Gibson's negative result on exponential stabilization, but we note that the original contractive system (A, B) is not strongly stable (see Example 3.2.2). Therefore the abovementioned result does not apply and it is possible to exponentially stabilize the system via compact fedback.

Also, note that $S(t)$ is again contractive due to the fact that BB^* and Q commute. $\triangle$

Example 4.3.2 Let $H = \overline{\text{span}\{\phi_n\}_{n=1}^{\infty}}$. In Example 3.3.2 we have shown that the operator

$$Ax = \sum_{n=1}^{\infty} -\frac{1}{n}[x,\phi_n]\phi_n$$

generates a strongly stable contractive semigroup $T(t)$. Then, in Example 4.2.2, we showed that with the control $u = -B^*x$ where

$$Bx = \sum_{n=1}^{\infty} \sqrt{\frac{2}{n}}[x,\phi_n]\phi_n \ ,$$

the closed loop operator $A - BB^*$ also generates a strongly but not exponentially stable contractive semigroup $Z(t)$.

Note that since

$$\lim_{n\to\infty} \sqrt{\frac{2}{n}} = 0 \ ,$$

B is a compact operator and $-B^*$ can not exponentially stabilize the strongly stable system (A, B).

The SSRE (4.17) admits the nonnegative solution

$$Qx = \frac{\sqrt{5}-1}{2} x \ .$$

Since the operator $-B^*Q$ is compact, the control $u(t) = -B^*Qx(t)$ can not exponentially stabilize the system either. To see this, we observe that the semigroup generated by $A - BB^*Q$ is given by

$$S(t)x = \sum_{n=1}^{\infty} e^{-\frac{\sqrt{5}}{n} t}[x, \phi_n]\phi_n$$

which is only strongly stable. Also note that the operators BB^* and Q commute, hence $S(t)$ is contractive. $\triangle$

Chapter 5

Conclusions

5.1 Results

In this dissertation we have investigated strong stability of distributed systems. Specifically, we have used three different approaches to find conditions for a C_0 (mainly contractive) semigroup to be strongly stable. Each one resulted in a different characterization. Based on these characterizations, we have studied the problem of feedback stabilization.

The theory of dilations of contractions and contractive semigroups was well developed in [18, 25, 51]. It was applied to scattering theory by Lax and Phillips [36]. Here we have applied the dilations theory to strong stability of contractive systems and showed that it results in a characterization of strongly stable contractive semigroups in terms of the left shift semigroup. Left shift semigroup is a canonical example of a strongly stable contractive semigroup. Via the dilations approach we have proved a sufficient condition for a contractive semigroup to be strongly stable but not exponentially stable. Although this approach results in (so far the only

known) necessary and sufficient condition for strong stability of a contractive semigroup, the results are not very suitable for application to control problems. This is mainly because it is generally difficult to compute the dilation of a given semigroup. In real-life problems even the semigroup asociated with the distributed system may be difficult to compute analytically. In such cases, the theory of dilations may still be applied via the notion of a co-generator for contractive semigroups.

A main contribution of this dissertation has been the development of a decomposition for the state space which identifies the strongly stable and unstable subspaces of a contractive semigroup. A similar approach has previously been used in the case of weak stability and resulted in a necessary and sufficient condition for weak stabilization [7]. This has been our motive. The decomposition developed here provides a clear picture of the structure of the state space with regard to strong stability. Based on the decomposition we have proved several conditions for strong stability of contractive semigroups.

Extension of Lyapunov's theorem to infinite dimensional Hilbert space was done by Datko [11, 12]. Datko proved that the existence of a positive (operator) solution to the infinite dimensional version of the Lyapunov equation is necessary and sufficient for exponential stability of the semigroup associated with the equation. Here, we have extended Lyapunov's theorem further to weak and strong stabilities of uniformly bounded semigroups. The uniform boundedness assumption is not a restrictive one since a (weakly, strongly or exponentially) stable semigroup is always uniformly bounded. By replacing the right hand side of the Lyapunov equation with a new norm which is weaker than the original norm, we proved that the existence of a nonnegative solution is sufficient for weak stability. In the same way, if the new

norm is stronger the same result holds for strong stability.

By applying the above characterizations to a distributed system $\dot{x} = Ax + Bu$ with the control $u = Fx$, one can obtain conditions for strong stabilization. In our case we have taken $F = -B^*$. The feedback $-B^*$ is robust in the sense that it does not depend on the system characteristics as reflected in the operator A. Also, no operator equations need to be solved in determining the feedback. Benchimol showed that the same feedback weakly stabilizes the system if weakly unstable states are controllable [7]. By applying the state space decomposition developed for strong stability we have proved a similar necessary and sufficient condition for strong stabilizability by $-B^*$. Application of the extensions of the Lyapunov equation also resulted in sufficient conditions for weak, strong and exponential stabilizations by the same feedback. Finally, we have proved an existence result concerning a linear quadratic regulator problem for contractive systems and the associated steady state Riccati equation. This fact was not pointed out before.

5.2 Future Directions

In this dissertation we have focused mainly on contractive systems. To our knowledge, there is still no result which gives a necessary and sufficient condition for a general C_0 semigroup to be strongly stable. For contractive semigroups, application of the theory of dilations resulted in a necessary and sufficient condition for strong stability. However, the result is not very suitable for practical applications. The first step for future work may be developing a more practical characterization for strongly stable contractive semigroups. The state space decomposition developed

in Chapter 3 also suggests a topic for further research: Identifying those contractive semigroups for which the subspace M is trivial or characterizing the class of contractive semigroups for which the operator C is a projection (see Theorem 3.2.2 for definitions).

The extensions of the Lyapunov equation gave sufficient conditions for weak and strong stabilities. We have only modified the right hand side of the Lyapunov equation. Further extensions need to be investigated in order to find necessary and sufficient conditions for weak and strong stabilities.

For strong stabilization of contractive systems we have only considered the feedback $-B^*$. A major consideration was preserving contractivity. However, any feedback operator F such that Re.$[BFx, x] \leq 0$ will preserve contractivity. Further research may involve application of the characterizations developed in Chapter 3 to more general feedbacks.

References

[1] Akhiezer, N. I., Glazman, I. M., *Theory of Linear Operators in Hilbert Space, Vol. I*, Frederick Ungar, New York, 1961.

[2] Akhiezer, N. I., Glazman, I. M., *Theory of Linear Operators in Hilbert Space, Vol. II*, Frederick Ungar, New York, 1963.

[3] Balakrishnan, A. V., *Applied Functional Analysis*, Springer Verlag, New York, 1981.

[4] Balakrishnan, A. V., Strong Stabilizability and the Steady State Riccati Equation, Appl. Math. Optim., Vol. 7, pp 335-345, 1981.

[5] Benchimol, C. D., A Note on Weak Stabilizability of Contraction Semigroups, SIAM J. Control & Optim., Vol. 16, No. 3, pp 373-379, 1978.

[6] Benchimol, C. D., Feedback Stabilizability in Hilbert Spaces, Appl. Math. Optim., Vol. 4, pp 225-248, 1978.

[7] Benchimol, C. D., *The Stabilizability of Infinite Dimensional Linear Time Invariant Systems*, Ph.D. Dissertation, UCLA, 1977.

[8] Bognar, Janos, *Indefinite Inner Product Spaces*, Springer Verlag, Heidelberg, 1974.

[9] Curtain, R. F., Pritchard, A. J., *Infinite Dimensional Linear System Theory*, Lecture Notes in Control and Information Sciences, No. 8, Springer Verlag, Berlin, 1978.

[10] Daleckii, Ju. L., Krein, M. G., *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc., Providence, RI, 1974.

[11] Datko, R., An Extension of a Theorem of A. M. Lyapunov to Semigroups of Operators, J. Math. Anal. & Appl., Vol. 24, pp 290-295, 1968.

[12] Datko, R., Extending a Theorem of A. M. Liapunov to Hilbert Space, J. Math. Anal. & Appl., Vol. 32, pp 610-616, 1970.

[13] Datko, R., A Linear Control Problem in an Abstract Hilbert Space, J. Diff. Eqn., Vol. 9, pp 346-359, 1971.

[14] Davies, E. B., *One-Parameter Semigroups*, Academic Press, New York, 1980.

[15] Dolecki, S., Russell, D. L., A General Theory of Observation and Control, SIAM J. Control & Optim., Vol. 15, No. 2, pp 185-220, 1977.

[16] Douglas, R. G., *Canonical Models*, Mathematical Surveys, Vol. 13, Amer. Math. Soc., 1974.

[17] Douglas, R. G., On Majorization, Factorization, and Range Inclusion of Operators on Hilbert Space, Proc. Amer. Math. Soc. Vol. 17, No. 2, pp 413-425, 1966.

[18] Fillmore, P. A., *Notes on Operator Theory*, Van Nostrand Reinhold, New York, 1970.

[19] Fillmore, P. A., Williams, J. P., On Operator Ranges, Advances in Mathematics, Vol. 7, pp 254-281, 1971.

[20] Foguel, S. R., Powers of a Contraction in Hilbert Space, Pac. J. Math., Vol. 13, pp 551-562, 1963.

[21] Gearhart, L., Spectral Theory for Contraction Semigroups on Hilbert Space, Trans. Amer. Math. Soc., Vol. 236, pp 385-394, 1978.

[22] Gibson, J. S., A Note on Stabilization of Infinite Dimensional Linear Oscillators by Compact Feedback, SIAM J. Control & Optim., Vol. 18, No. 3, pp 311-316, 1980.

[23] Goldstein, J. A., *Semigroups of Linear Operators and Applications*, Oxford Mathematical Monographs, Oxford University Press, New York, 1985.

[24] Halmos., P. R., *A Hilbert Space Problem Book*, Van Nostrand, Princeton, NJ, 1967.

[25] Halmos., P. R., *Invariant Subspaces*, Coloquio Brasileiro de Matematica, Brazil, 1969.

[26] Helson, H., *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[27] Hille, E., *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloquium, Vol. 31, 1948.

[28] Hille, E., *Methods in Classical and Functional Analysis*, Addison Wesley, 1972.

[29] Hille, E., Phillips, R. S., *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloquium, Vol. 31, 1957.

[30] Kato, T., *Quadratic Forms in Hilbert Spaces and Asymptotic Perturbation Series*, Technical Report for Office of Ordnance Research, U. S. Army, Univ. of California, Berkeley, CA, April 1955.

[31] Kato, T., *Perturbation Theory for Linear Operators*, 2nd Ed., Springer Verlag, Berlin, 1980.

[32] Kantorovich, L. V., Akilov, G. P., *Functional Analysis*, 2nd Ed., Pergamon Press, Oxford, 1982.

[33] Krein, S. G., *Linear Differential Equations in Banach Space*, Translation of Math. Monographs, Vol. 29, Amer. Math. Soc., Providence, RI, 1971.

[34] Krein, S. G., *Linear Equations in Banach Spaces*, Birkhauser Boston Inc., 1982.

[35] LaSalle, J. P., Stability Theory for Ordinary Differential Equations, J. Diff. Eqn., Vol. 4, pp 57-65, 1968.

[36] Lax, P. D., Phillips, R. S., *Scattering Theory*, Academic Press, New York, 1967.

[37] Lax, P. D., Phillips, R. S., The Translation Representation Theorem,in *Integral Equations and Operator Theory*, Vol. 4/3, Birkhauser Verlag, Switzerland, 1981.

[38] Levan, N., The Stabilizability Problem: A Hilbert Space Operator Decomposition Approach, IEEE Trans. Circuits & Systems, Vol. CAS-25, No. 9, pp 721-727, 1978.

[39] Levan, N., On Some Relationships Between the LaSalle Invariance Principle and the Nagy-Foias Canonical Decomposition, J. Math. Anal. & Appl., Vol. 77, pp 493-504, 1980.

[40] Levan, N., Controllability, *-Controllability and Stabilizability, J. Diff. Eqn., Vol. 38, No. 2, pp 61-79, 1980.

[41] Levan, N., Stabilization by Dissipative Feedback Systems, Proc. 23rd IEEE Conf. Decision & Control, Las Vegas, pp 1176-1179, 1984.

[42] Levan, N., Stabilization by the Feedbacks $-B^*$ and $-B^*P$, in System Modelling and Optimization, Proc. 12th IFIP Conf., Budapest, Hungary, Springer Verlag, No. 84, 1985.

[43] Levan, N., Strong Stability of Quasi-affine Transforms of Contraction Semigroups and the Steady State Riccati Equation, J. Optim. Theory & Appl., Vol. 45, No. 3, pp 397-405, 1985.

[44] Levan, N., Approximate Stabilizability via the Algebraic Riccati Equation, SIAM J. Control & Optim., Vol. 23, No. 1, pp 153-160, 1985.

[45] Levan, N., Rigby, L., On the Reduction of a Contraction Semigroup to a Completely Non-Unitary Semigroup, J. Math. Anal. & Appl., Vol. 67, pp 1-11, 1979.

[46] Levan, N., Rigby, L., Strong Stabilizability of Linear Contractive Control Systems on Hilbert Space, SIAM J. Control & Optim., Vol. 17, No. 1, pp 23-35, 1979.

[47] Louis, J. C., Wexler, D., On Exact Controllability in Hilbert Spaces, J. Diff. Eqn., Vol. 49, pp 258-269, 1983.

[48] Miyaji, W., *Strong and Weak Stabilizability: Lyapunov Type Approaches*, Ph.D. Dissertation, UCLA, 1988.

[49] Sz.-Nagy, B., Ed.,*Hilbert Space Operators and Operator Algebras*, Colloquia Mathematica Societatis Janos Bolyai, North Holland Publishing Co., 1970.

[50] Sz.-Nagy, B., *Unitary Dilations of Hilbert Space Operators and Related Topics*, Amer. Math. Soc., Regional Conf. Series in Math., No. 19, Rhode Island, 1974.

[51] Sz.-Nagy, B., Foias, C., *Harmonic Analysis of Operators on Hilbert Space*, Amer. Elsevier, New York, 1970.

[52] O'Brien, R. E. Jr., Contraction Semigroups, Stabilization and the Mean Ergodic Theorem, Proc. Amer. Math. Soc., Vol. 71, No. 1, 1978.

[53] O'Brien, R. E. Jr., Weakly Stable Operators, J. Math. Anal. & Appl., Vol. 70, pp 170-179, 1979.

[54] Pazy, A., On the Applicability of Lyapunov's Theorem in Hilbert Space, SIAM J. Math. Anal., Vol. 3, pp 291-294, 1972.

[55] Pazy, A., *Semigroups of Linear Operators & Applications to Partial Differential Equations*, Springer Verlag, Vol. 44, New York, 1983.

[56] Pritchard, A. J., Zabczyk, J., Stability and Stabilizability of Infinite Dimensional Systems, SIAM REVIEW, Vol. 23, No. 1, pp 25-52, 1981.

[57] Ray, H.W., Lainiotis, D. G., Ed.'s, *Distributed Parameter Systems*, Marcel Dekker Inc., New York, 1978.

[58] Riesz, F., Sz.-Nagy, B., *Functional Analysis*, Frederick Ungar, New York, 1955.

[59] Royden, H. L., *Real Analysis*, 2nd Ed., MacMillan Publishing Co., New York, 1968.

[60] Russell, D. L. Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions, SIAM Review, Vol. 20, No. 4, pp 639-739, October 1978.

[61] Showalter, R. E., *Hilbert Space Methods for Partial Differential Equations*, Pitman Publishing, London, 1979.

[62] Slemrod, M., The Linear Stabilization Problem in Hilbert Space, J. Func. Anal., Vol. 11, pp 334-345, 1972.

[63] Slemrod, M., A Note on Complete Controllability and Stabilizability of Linear Control Systems in Hilbert Space, SIAM J. Control, Vol. 12, No. 3, 1974.

[64] Stone, M. H., *Linear Transformations in Hilbert Space*, Amer. Math. Soc. Colloquium Publications, Vol. 15, New York, 1932.

[65] Triggiani, R., Pathological Asymptotic Behavior of Control Systems in Banach Space, J. Math. Anal. & Appl., Vol. 49, No. 2, pp 411-429, 1975.

[66] Triggiani, R., On the Stabilizability Problem In Banach Space, J. Math. Anal. & Appl., Vol. 52, pp 383-403, 1975.

[67] Triggiani, R., A^{ϵ}-Bounded, Finite Rank Perturbations of S.C. Group Generators A: Counterexamples to Generation and to Another Condition for Well-Posedness, in *Infinite Dimensional Systems*, pp 227-253, Proceedings, Retzhof, Austria, 1983, Springer Verlag, No. 1076, 1984.

[68] Triggiani, R., Lack of Uniform Stabilization for Non-Contractive Semigroups Under Compact Perturbation, to appear.

[69] Van Casteren, J. A., *Generators of Strongly Continuous Semigroups*, Pitman Publishing Ltd., London, 1985.

[70] Yosida, K., *Functional Analysis*, Springer Verlag, Berlin, 1980.

[71] Zabczyk, J., Remarks on The Algebraic Riccati Equation in Hilbert Space, Appl. Math. Optim., Vol. 2, No. 3, pp 383-403, 1976.

[72] Zabczyk, J., Remarks on the Control of Discrete Time Distributed Parameter Systems, SIAM J. Control, Vol. 12, pp 721-735, 1974.